MS221 EBC

The Open University

EXPLORING MATHEMATICS

Exercise Booklet C

Exercise Booklet

C

SUP 68143 1

Contents

The exercises in this booklet are intended to give further practice, should you require it, in handling the main mathematical ideas in each chapter of MS221, Block C. The exercises are ordered by chapter and section, and are numbered accordingly: for example, Exercise 3.2 for Chapter C1 is the second exercise on Section 3 of that chapter.

Exercises for Chapter C1

Section 1

Exercise 1.1

(a) Consider the function $k(x) = x^2 + 3x$.

(i) Show that if $h \neq 0$, then

$$\frac{k(x+h) - k(x)}{h} = 2x + h + 3.$$

(ii) Hence determine $k'(x)$.

(b) Consider the function $f(x) = \dfrac{1}{x^3}$ $(x \neq 0)$.

(i) Show that if $h \neq 0$ and $x + h \neq 0$, then

$$\frac{f(x+h) - f(x)}{h} = -\frac{(3x^2 + 3xh + h^2)}{(x+h)^3 x^3}.$$

(ii) Hence determine $f'(x)$.

Exercise 1.2

(a) Given that $f(x) = x^{4/3}$ write down

$f'(x), f''(x)$ and $f^{(3)}(x)$.

(b) Given that $y = \cos x$, write down

$$\frac{dy}{dx}, \quad \frac{d^2y}{dx^2}, \quad \frac{d^3y}{dx^3} \text{ and } \frac{d^4y}{dx^4}.$$

Section 2

Exercise 2.1

(a) Given that $f(x) = 4\sqrt{x} - 3\sin x$ $(x > 0)$, find $f'(x)$.

(b) Given that $g(t) = 2e^t + 3\ln t$ $(t > 0)$, find $g'(t)$.

Exercise 2.2

Use the Product Rule to differentiate each of the following functions.

(a) $f(x) = (x^3 - 8x^{4/3} + 2)\cos x$

(b) $g(t) = (2t^4 + \sin t)\tan t$

(c) $h(x) = \left(5x^7 - x^2 + 5 - \dfrac{2}{x}\right) e^x$

(d) $f(\theta) = \theta^6 \ln \theta$

Exercise 2.3

Use the Quotient Rule to differentiate each of the following functions.

(a) $f(x) = \dfrac{x^7 + 5x^3 + 1}{x^4 + 1}$

(b) $g(x) = \dfrac{4x + e^x}{x^8 - 2}$

(c) $h(t) = \dfrac{1 + \cos t}{\sin t}$

Exercise 2.4

Differentiate each of the following functions.

(a) $k(x) = (x^2 - 3x + 4)(x^2 + 2x - 1)$

(b) $g(t) = \ln t \cos(3t)$

(c) $h(\theta) = \dfrac{\sin\theta}{\theta^2 + 2\theta - 1}$

(d) $f(x) = \dfrac{x^2 + 2x - 1}{x^2 - 3x + 4}$

(e) $k(x) = \dfrac{x^3 \sin x}{e^x}$

Exercise 2.5

Use the Composite Rule to differentiate each of the following functions.

(a) $k(x) = \cos(\sqrt{x})$

(b) $k(x) = \sqrt{\cos x}$

(c) $k(x) = e^{x^2/2}$

(d) $k(x) = \sin(x^3)$

(e) $k(x) = \tan(\ln x)$

Exercise 2.6

Differentiate each of the following functions.

(a) $f(t) = \cos^2(3t)\ln(5t)$

(b) $f(x) = \dfrac{e^{2x}}{(x^2+1)^2}$

(c) $f(x) = \sin\left((x^2 + 4)e^{3x}\right)$

(d) $f(t) = e^{t^3/\sin t}$

Exercise 2.7

(a) Use the Inverse Rule to find the derivative of $f(x) = \arcsin(2x)$, for $-\frac{1}{2} < x < \frac{1}{2}$.

(b) (i) Given that $y = \arctan(3x)$, use the Inverse Rule to find dy/dx.

(ii) Confirm your answer by using the derivative of arctan and the Chain Rule.

Section 3

Exercise 3.1

Sketch the graph of each of the following functions.

(a) $f(x) = \dfrac{3x-8}{x-4}$

(b) $k(x) = \dfrac{x^2+1}{x^2-1}$

(c) $h(x) = x^3 - 9x$

(d) $g(x) = (x^2-5)e^{x/2}$ (You may assume that $g(x) \to 0$ as $x \to -\infty$.)

Section 4

Exercise 4.1

(a) Consider the function $f(x) = x^3 + 3x^2 - 2$.

(i) By evaluating $f(-3)$ and $f(-2)$, show that the equation $f(x) = 0$ has a solution in the interval $(-3, -2)$.

(ii) Show that for this function f the Newton–Raphson formula in equation (4.1) can be expressed as

$$x_{n+1} = \frac{2x_n^3 + 3x_n^2 + 2}{3x_n(x_n+2)} \quad (n = 0, 1, 2, \ldots).$$

(iii) Use your calculator to find the first five terms of the sequence x_n, when the initial term is $x_0 = -2.5$.

(iv) Check that the final term calculated in part (iii) is indeed a good approximation to a solution of $f(x) = 0$.

(b) Consider the function $f(x) = e^x - 3x$.

(i) By evaluating $f(0)$ and $f(1)$, show that the equation $f(x) = 0$ has a solution in the interval $(0, 1)$.

(ii) Show that for this function f the Newton–Raphson formula in equation (4.1) can be expressed as

$$x_{n+1} = \frac{e^{x_n}(x_n - 1)}{e^{x_n} - 3} \quad (n = 0, 1, 2, \ldots).$$

(iii) Use your calculator to find the first five terms of the sequence x_n, when the initial term is $x_0 = 0.5$.

(iv) Check that the final term calculated in part (iii) is indeed a good approximation to a solution of $f(x) = 0$.

Exercise 4.2

(a) Consider the function $f(x) = x^4 - 3x - 3$.

(i) By evaluating $f(1)$ and $f(2)$, show that the equation $f(x) = 0$ has a solution in the interval $(1, 2)$.

(ii) Show that for this function f the Newton–Raphson formula in equation (4.1) can be expressed as

$$x_{n+1} = \frac{3x_n^4 + 3}{4x_n^3 - 3} \quad (n = 0, 1, 2, \ldots).$$

(iii) Use your calculator to find the first five terms of the sequence x_n, when the initial term is $x_0 = 1.5$.

(iv) Check that the final term calculated in part (iii) is indeed a good approximation to a solution of $f(x) = 0$.

(v) Find a value of x_0 for which the Newton–Raphson method fails for this function f.

(b) Consider the function $f(x) = \frac{1}{2}x + \cos x$.

(i) By evaluating $f(-2)$ and $f(-1)$, show that the equation $f(x) = 0$ has a solution in the interval $(-2, -1)$.

(ii) Show that for this function f the Newton–Raphson formula in equation (4.1) can be expressed as

$$x_{n+1} = \frac{x_n \sin x_n + \cos x_n}{\sin x_n - \frac{1}{2}} \quad (n = 0, 1, 2, \ldots).$$

(iii) Use your calculator to find the first five terms of the sequence x_n, when the initial term is $x_0 = -1.5$.

(iv) Check that the final term calculated in part (iii) is indeed a good approximation to a solution of $f(x) = 0$.

(v) Find a value of x_0 for which the Newton–Raphson method fails for this function f.

(vi) Generalise your result in part (v) to give a number of initial values x_0 for which the Newton–Raphson method fails for this function f.

Exercises for Chapter C2

Section 1

Exercise 1.1

Use the table in the Handbook and the Sum and Constant Multiple rules to find each of the following indefinite integrals.

(a) $\displaystyle\int (3x^2 + \cos(4x))\,dx$

(b) $\displaystyle\int (e^{2t} - \sin(5t))\,dt$

(c) $\displaystyle\int \frac{4}{1+t^2}\,dt$

(d) $\displaystyle\int \left(\frac{e^{3x}+e^{2x}}{e^x}\right)\,dx$

Exercise 1.2

Evaluate the following definite integrals, giving your answers to 4 decimal places.

(a) $\displaystyle\int_1^2 (3x^4 - e^{2x})\,dx$

(b) $\displaystyle\int_{-\pi}^{\pi} \cos\left(\tfrac{1}{5}t\right)\,dt$

(c) $\displaystyle\int_{-1/2}^{1/2} \frac{1}{\sqrt{1-u^2}}\,du$

(d) $\displaystyle\int_0^1 e^{t/2}\,dt$

Exercise 1.3

* (a) (i) Explain why the graph of $f(x) = x^{10}$ is above the x-axis for all values of x in the interval $[-3,-2]$.

(ii) Find the area under the graph of $f(x) = x^{10}$ from $x=-3$ to $x=-2$, correct to the nearest integer.

(b) (i) Explain why the graph of $y = \sin(5x)$ is above the x-axis for all values of x in the interval $[\pi/15, 2\pi/15]$.

(ii) Find the area under the graph of $y = \sin(5x)$ from $x = \pi/15$ to $x = 2\pi/15$.

(c) (i) Find the values of $\displaystyle\int_2^3 (e^{x/2}-1)\,dx$ and $\displaystyle\int_{-1}^2 (e^{x/2}-1)\,dx$, correct to 4 decimal places.

(ii) Which of the integrals above represents the area bounded by the graph of $y = e^{x/2}-1$, the x-axis and the stated limits for x?

(iii) Calculate the area bounded by this graph, the x-axis, $x=-1$ and $x=3$, correct to 4 decimal places.

(d) Sketch the graph of the function $f(x) = \cos(2x)$ between $x=-\pi/2$ and $x=\pi/2$. Using the fact that the area bounded by the graph, the x-axis and the given limits for x is 4 times the area bounded by the graph, the x-axis, $x=0$ and $x=\pi/4$, show that the area bounded by the graph and the x-axis between $x=-\pi/2$ and $x=\pi/2$ is 2.

(e) Sketch the curve $y^2 = 4x$ from $x=0$ to $x=9$. Using your sketch to help you, find the area bounded by the curve and the line $x=9$.

Section 2

Exercise 2.1

Use integration by parts to find each of the following indefinite integrals.

(a) $\displaystyle\int x\cos(3x)\,dx$

(b) $\displaystyle\int 2x\sin\left(\tfrac{1}{5}x\right)\,dx$

(c) $\displaystyle\int x\ln(5x)\,dx$

Exercise 2.2

Use integration by parts to find each of the following definite integrals correct to 3 decimal places.

(a) $\displaystyle\int_0^1 xe^{4x}\,dx$

(b) $\displaystyle\int_{-\pi/4}^{\pi/4} x\sin(2x)\,dx$

(c) $\displaystyle\int_1^2 x^2\ln(4x)\,dx$

Exercise 2.3

Find the following integrals.

(a) $\displaystyle\int x^2\sin\left(\tfrac{1}{2}x\right)\,dx$

(b) $\displaystyle\int e^{x/3}\cos(4x)\,dx$

(c) $\displaystyle\int e^{-2x}\sin(5x)\,dx$

(d) $\displaystyle\int x^2\ln x\,dx$

Exercise 2.4

Evaluate the following integrals, correct to 4 decimal places.

(a) $\displaystyle\int_0^1 x^2 e^{5x}\,dx$

(b) $\displaystyle\int_0^{\pi/4} e^{x/2}\cos(2x)\,dx$

(c) $\displaystyle\int_1^4 x^2 \ln x\,dx$

Section 3

Exercise 3.1

Use integration by substitution to find the following integrals.

(a) $\displaystyle\int x^4 e^{x^5}\,dx$, taking $u = x^5$

(b) $\displaystyle\int x^2 \sin(x^3)\,dx$, taking $u = x^3$

(c) $\displaystyle\int x^2(1-2x^3)^9\,dx$, taking $u = 1-2x^3$

(d) $\displaystyle\int \frac{x^5}{2-x^6}\,dx$, taking $u = 2-x^6$

Exercise 3.2

Use integration by substitution to evaluate the following integrals, giving your answers correct to 4 decimal places.

(a) $\displaystyle\int_0^1 x(2x^2+1)^{1/2}\,dx$, taking $u = 2x^2+1$

(b) $\displaystyle\int_0^1 xe^{x^2+3}\,dx$, taking $u = x^2+3$

(c) $\displaystyle\int_0^{1/\sqrt{3}} \frac{x}{1+9x^4}\,dx$, taking $u = \arctan(3x^2)$

(d) $\displaystyle\int_0^{\pi/2} \frac{\cos x \sin x}{1+\sin^2 x}\,dx$, taking $u = 1+\sin^2 x$

Exercise 3.3

Use integration by backwards substitution to find the following integrals.

(a) $\displaystyle\int \frac{x}{(3x-1)^4}\,dx$, taking $x = \frac{1}{3}(u+1)$, where $u = 3x-1$

(b) $\displaystyle\int \frac{x^2}{(x+1)^{1/2}}\,dx$, taking $x = u^2-1$, where $u = (x+1)^{1/2}$

(c) $\displaystyle\int \frac{\ln x}{x}\,dx$, taking $x = e^u$, where $u = \ln x$

(d) $\displaystyle\int \frac{1}{(1+x^2)^{3/2}}\,dx$, taking $x = \tan u$, where $u = \arctan x$

Exercise 3.4

By choosing an appropriate substitution, find each of the following integrals.

(a) $\displaystyle\int x^4 \sin(x^5)\,dx$

(b) $\displaystyle\int \cos^7(2x)\sin(2x)\,dx$

(c) $\displaystyle\int x\sqrt{x-1}\,dx$

(d) $\displaystyle\int e^x \cos(e^x)\,dx$

Exercise 3.5

Find the following integrals, using any appropriate method.

(a) $\displaystyle\int x^2 \sin(4x^3)\,dx$

(b) $\displaystyle\int x \sin(6x)\,dx$

(c) $\displaystyle\int \frac{x^3}{(8-x^4)^6}\,dx$

Exercise 3.6

Evaluate the following integrals, using any appropriate method, giving your answers to 4 decimal places.

(a) $\displaystyle\int_1^2 \frac{x+2}{(x+1)^2}\,dx$

(b) $\displaystyle\int_0^{\pi/4} \sec^4 x \tan x\,dx$

(c) $\displaystyle\int_{-\pi/6}^{\pi/6} x\cos(6x)\,dx$

Section 4

Exercise 4.1

(a) Find the volume of revolution obtained when the region under the graph of $y = 1/x$, from $x = \frac{1}{2}$ to $x = 1$, is rotated about the x-axis.

(b) Find the volume of revolution obtained when the ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$ is rotated about the x-axis.

(This solid of revolution is the same as that obtained by rotating the graph of $f(x) = 2\sqrt{1-x^2/9}$ about the x-axis.)

(c) Use integration by substitution to find the volume of revolution obtained when the region under the graph of $y = \sqrt{x}(1+x)^{1/3}$, from $x = 0$ to $x = 1$, is rotated about the x-axis, giving your answer correct to 4 decimal places.

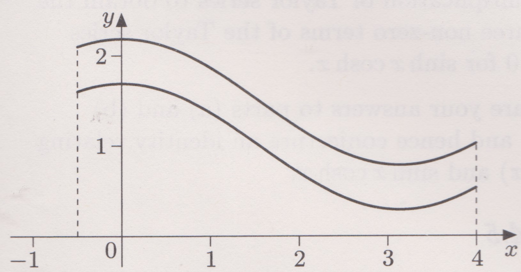
(d) Use integration by parts to find the volume of revolution obtained when the region under the graph of $y = xe^{-x}$, from $x = 1$ to $x = 2$, is rotated about the x-axis, giving your answer correct to 4 decimal places.

(e) Find the volume of revolution obtained when the region under the graph of $y = \sin x$, from $x = 0$ to $x = \pi$, is rotated about the x-axis, giving your answer correct to 4 decimal places.

Exercise 4.2

A miniature perfume bottle is 4.5 cm high and the outside shape is formed by the curve with equation $y = 1.5 + \dfrac{1}{\sqrt{2}} \cos x$ between $x = -\frac{1}{2}$ and $x = 4$. It has a circular cross-section.

The thickness of the material from which the bottle is made is uniform and measures 0.5 cm; see figure below.



Explain why the internal volume V of the bottle can be expressed by

$$V = \pi \int_0^4 \left(1 + \frac{1}{\sqrt{2}} \cos x\right)^2 dx.$$

Hence calculate how much perfume the bottle can hold, to the nearest $0.01\,\text{cm}^3$.

Exercises for Chapter C3

Section 1

Exercise 1.1

Find the linear Taylor polynomial about 0 for each of the functions below. Use the polynomial to find an approximation for $f(0.02)$, and use your calculator to find the value of the associated remainder to eight decimal places.

(a) $f(x) = e^{-x}$ (b) $f(x) = (4 - x)^{1/2}$ $(x < 4)$

Exercise 1.2

Find the linear Taylor polynomial about 1 for each of the functions below.

(a) $f(x) = e^{-x}$ (b) $f(x) = \dfrac{1}{1 + x}$

Exercise 1.3

(a) Use the linear Taylor polynomial about 1 from Exercise 1.2(b) above to find an approximate value for the reciprocal of 2.01. Use your calculator to find the value of the associated remainder to eight decimal places.

(b) Find the linear Taylor polynomial about 0 for $f(x) = \dfrac{1}{(1 + x)^3}$. Use your answer to find an approximate value for the reciprocal of 1.01^3. Use your calculator to find the value of the associated remainder to eight decimal places.

Exercise 1.4

Find the quadratic Taylor polynomial about 0 for each of the functions below. Use the polynomial to find an approximation for $f(0.01)$, and use your calculator to find the value of the associated remainder to eight decimal places.

(a) $f(x) = e^{x/2}$

(b) $f(x) = x \cos x$

Section 2

Exercise 2.1

Find the quartic Taylor polynomial about 0 for each of the functions below.

(a) $f(x) = \cos(2x)$

(b) $f(x) = \ln\left(\dfrac{1}{1 + x}\right)$

(c) $f(x) = \sqrt{1 - x}$

Exercise 2.2

(a) Find the quintic Taylor polynomial about π for the function $f(x) = \cos x$.

(b) Find the quintic Taylor polynomial about e for the function $f(x) = \ln x$.

Exercise 2.3

(a) Given that the Taylor polynomial of degree n about 0 for the function $f(x) = \ln(1 + x)$ is

$$p_n(x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \cdots + \frac{(-1)^{n+1}}{n} x^n,$$

for $n = 1, 2, 3, \ldots$, calculate the value of $\ln(1.05)$ to four decimal places.

(b) Given that the Taylor polynomial of degree $2n$ about 0 for the function $f(x) = \cos(2x)$ can be expressed as

$$p_{2n}(x) = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \cdots + (-1)^n \frac{(2x)^{2n}}{(2n)!},$$

for $n = 0, 1, 2 \ldots$, calculate the value of $\cos(0.2)$ to four decimal places.

Section 3

Exercise 3.1

(a) Find the Taylor series about 0 for the function $f(x) = e^{-x}$.

(b) Find the Taylor series about 1 for the function $f(x) = 1/x$.

Exercise 3.2

Use the binomial series to find the first five terms of the Taylor series about 0 for each of the following functions.

(a) $(1+x)^{-4}$ (b) $(1+x)^{1/3}$

Exercise 3.3

Using the appropriate series from Exercise 3.2, find the value of each of the following numbers to three decimal places.

(a) $(0.99)^{-4}$ (b) $(0.9)^{1/3}$

Section 4

Exercise 4.1

Using the standard Taylor series about 0, find the Taylor series about 0 for each of the following functions, giving the first four non-zero terms. For each series state a range of validity.

(a) $\sin(x^2)$ (b) $e^{x/2}$

(c) $\dfrac{1}{1-2x}$ (d) $(1+x^2)^{3/2}$

Exercise 4.2

Using standard Taylor series about 0, find the Taylor series about 0 for each of the following functions, giving the first four non-zero terms. For each series state a range of validity.

(a) $e^x + \cos x$

(b) $\ln\left(\dfrac{1-x}{1+x}\right)$

(*Hint*: Use $\ln\left(\dfrac{1-x}{1+x}\right) = \ln(1-x) - \ln(1+x)$.)

(c) $(1-x)\sin x$

(d) $(1+x)^2 e^x$

Exercise 4.3

(a) Use the standard Taylor series about 0 for e^x to write down the Taylor series about 0 for e^{2x}.

(b) Use $e^{2x} = e^x \times e^x$ and multiplication of the standard Taylor series about 0 for e^x to confirm the first four terms of your answer to part (a).

Exercise 4.4

(a) The Taylor series about 0 for $\sinh x$ is

$$\sinh x = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots.$$

Use this series to write down the first three non-zero terms of the Taylor series about 0 for $\sinh(2x)$.

(b) The Taylor series about 0 for $\cosh x$ is

$$\cosh x = 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots.$$

Use multiplication of Taylor series to obtain the first three non-zero terms of the Taylor series about 0 for $\sinh x \cosh x$.

(c) Compare your answers to parts (a) and (b) above, and hence conjecture an identity relating $\sinh(2x)$ and $\sinh x \cosh x$.

Exercise 4.5

(a) By differentiating the standard Taylor series about 0 for $\ln(1+x)$, find the Taylor series about 0 for $\dfrac{1}{1+x}$.

(b) Use the Taylor series about 0 for $\dfrac{1}{1+x}$ to find the Taylor series about 0 for $\dfrac{1}{1+4x^2}$.

(c) Use your answer to part (b) to find the first four non-zero terms of the Taylor series about 0 for $\arctan(2x)$. (You may assume that an antiderivative of $1/(1+4x^2)$ is $\frac{1}{2}\arctan(2x)$.)

For each series state a range of validity.

Exercise 4.6

(a) Use the binomial series to find the Taylor series about 0 for $\dfrac{1}{\sqrt{1-x^2}}$, giving the first four non-zero terms. State a range of validity for the series.

(b) Use your answer to part (a) to find the Taylor series about 0 for $\arccos x$, giving the first five non-zero terms.

Solutions for Chapter C1

Solution 1.1

(a) (i) We have, for $h \neq 0$

$$\begin{aligned}&\frac{k(x+h)-k(x)}{h}\\&=\frac{1}{h}\left((x+h)^2+3(x+h)-(x^2+3x)\right)\\&=\frac{1}{h}\left(x^2+2xh+h^2+3x+3h-x^2-3x\right)\\&=2x+h+3,\end{aligned}$$

as required.

(ii) By equation (1.2) we need to consider the quotient from part (i):

$$\lim_{h\to 0}(2x+h+3)=2x+3,$$

so

$$k'(x)=2x+3.$$

(b) (i) We have, for $h \neq 0$ and $x+h \neq 0$,

$$\begin{aligned}&\frac{f(x+h)-f(x)}{h}\\&=\frac{1}{h}\left(\frac{1}{(x+h)^3}-\frac{1}{x^3}\right)\\&=\frac{1}{h}\left(\frac{x^3-(x+h)^3}{(x+h)^3x^3}\right)\\&=\frac{1}{h}\left(\frac{x^3-(x^3+3x^2h+3xh^2+h^3)}{(x+h)^3x^3}\right)\\&=-\frac{(3x^2+3xh+h^2)}{(x+h)^3x^3},\end{aligned}$$

as required.

(ii) By equation (1.2) we need to consider the quotient from part (i):

$$\lim_{h\to 0}\left(-\frac{(3x^2+3xh+h^2)}{(x+h)^3x^3}\right)=\frac{-3x^2}{x^3x^3}=\frac{-3}{x^4},$$

so

$$f'(x)=-\frac{3}{x^4}.$$

Solution 1.2

(a) Here $f(x)=x^{4/3}$, so $f'(x)=\frac{4}{3}x^{1/3}$, $f''(x)=\frac{4}{9}x^{-2/3}$ and $f^{(3)}(x)=-\frac{8}{27}x^{-5/3}$.

(b) Here $y=\cos x$, so $\dfrac{dy}{dx}=-\sin x$, $\dfrac{d^2y}{dx^2}=-\cos x$, $\dfrac{d^3y}{dx^3}=\sin x$ and $\dfrac{d^4y}{dx^4}=\cos x$.

(The Constant Multiple Rule was used here several times.)

Solution 2.1

(a) By the Sum and Constant Multiple Rules, the function $f(x)=4\sqrt{x}-3\sin x$ has first derivative

$$\begin{aligned}f'(x)&=4\times\tfrac{1}{2}x^{-1/2}-3\cos x\\&=\frac{2}{\sqrt{x}}-3\cos x.\end{aligned}$$

(b) By the Sum and Constant Multiple Rules, the function $g(t)=2e^t+3\ln t$ has first derivative

$$g'(t)=2e^t+\frac{3}{t}\quad(t>0).$$

Solution 2.2

(a) Here $f(x)=(x^3-8x^{4/3}+2)\cos x$, so

$$\begin{aligned}f'(x)&=\left(3x^2-8\times\tfrac{4}{3}x^{1/3}\right)\cos x\\&\quad+(x^3-8x^{4/3}+2)(-\sin x)\\&=\left(3x^2-\tfrac{32}{3}x^{1/3}\right)\cos x-(x^3-8x^{4/3}+2)\sin x.\end{aligned}$$

(b) Here $g(t)=(2t^4+\sin t)\tan t$, so

$$g'(t)=(8t^3+\cos t)\tan t+(2t^4+\sin t)\sec^2 t.$$

(c) Here $h(x)=(5x^7-x^2+5-\dfrac{2}{x})e^x$, so

$$\begin{aligned}h'(x)&=(35x^6-2x+2x^{-2})e^x\\&\quad+\left(5x^7-x^2+5-\frac{2}{x}\right)e^x\\&=\left(5x^7+35x^6-x^2-2x+5-\frac{2}{x}+\frac{2}{x^2}\right)e^x.\end{aligned}$$

(d) Here $f(\theta)=\theta^6\ln\theta$, so

$$\begin{aligned}f'(\theta)&=6\theta^5\ln\theta+\theta^6\times\frac{1}{\theta}\\&=\theta^5(6\ln\theta+1).\end{aligned}$$

Solution 2.3

(a) Here $f(x)=\dfrac{x^7+5x^3+1}{x^4+1}$, so

$$\begin{aligned}f'(x)&=\frac{(x^4+1)(7x^6+15x^2)-(x^7+5x^3+1)(4x^3)}{(x^4+1)^2}\\&=\frac{3x^{10}+2x^6-4x^3+15x^2}{(x^4+1)^2}.\end{aligned}$$

(b) Here $g(x)=\dfrac{4x+e^x}{x^8-2}$, so

$$\begin{aligned}g'(x)&=\frac{(x^8-2)(4+e^x)-(4x+e^x)(8x^7)}{(x^8-2)^2}\\&=\frac{-28x^8-8+(x^8-8x^7-2)e^x}{(x^8-2)^2}.\end{aligned}$$

(c) Here $h(t) = \dfrac{1+\cos t}{\sin t}$, so

$$\begin{aligned} h'(t) &= \frac{\sin t(-\sin t) - (1+\cos t)\cos t}{\sin^2 t} \\ &= \frac{-\sin^2 t - \cos^2 t - \cos t}{\sin^2 t} \\ &= \frac{-1-\cos t}{\sin^2 t} \\ &= \frac{1}{\cos t - 1}, \end{aligned}$$

by using

$$\sin^2 t = 1 - \cos^2 t = (1-\cos t)(1+\cos t).$$

Solution 2.4

(a) Here $k(x) = (x^2-3x+4)(x^2+2x-1)$, so we use the Product Rule:

$$\begin{aligned} k'(x) &= (2x-3)(x^2+2x-1) \\ &\quad + (x^2-3x+4)(2x+2) \\ &= 2x^3+4x^2-2x-3x^2-6x+3 \\ &\quad + 2x^3-6x^2+8x+2x^2-6x+8 \\ &= 4x^3-3x^2-6x+11. \end{aligned}$$

(b) Here $g(t) = \ln t\cos(3t)$, so we use the Product Rule:

$$\begin{aligned} g'(t) &= \frac{1}{t}\cos(3t) + \ln t\,(-3\sin(3t)) \\ &= \frac{1}{t}\cos(3t) - 3\ln t\sin(3t). \end{aligned}$$

(c) Here $h(\theta) = \dfrac{\sin\theta}{\theta^2+2\theta-1}$, so we use the Quotient Rule:

$$\begin{aligned} h'(\theta) &= \frac{(\theta^2+2\theta-1)\cos\theta - \sin\theta(2\theta+2)}{(\theta^2+2\theta-1)^2} \\ &= \frac{(\theta^2+2\theta-1)\cos\theta - 2(\theta+1)\sin\theta}{(\theta^2+2\theta-1)^2}. \end{aligned}$$

(d) Here $f(x) = \dfrac{x^2+2x-1}{x^2-3x+4}$, so we use the Quotient Rule:

$$\begin{aligned} f'(x) &= \big((x^2-3x+4)(2x+2) \\ &\qquad -(x^2+2x-1)(2x-3)\big)/(x^2-3x+4)^2 \\ &= \frac{2x^3-4x^2+2x+8-2x^3-x^2+8x-3}{(x^2-3x+4)^2} \\ &= \frac{-5x^2+10x+5}{(x^2-3x+4)^2} \\ &= \frac{5(1+2x-x^2)}{(x^2-3x+4)^2}. \end{aligned}$$

(e) Here $k(x) = \dfrac{x^3\sin x}{e^x}$, so we use the Quotient and Product Rules:

$$\begin{aligned} k'(x) &= \frac{e^x(3x^2\sin x + x^3\cos x) - x^3\sin x(e^x)}{(e^x)^2} \\ &= \frac{e^x(3x^2\sin x + x^3\cos x - x^3\sin x)}{(e^x)^2} \\ &= \frac{x^2(3-x)\sin x + x^3\cos x}{e^x}. \end{aligned}$$

Solution 2.5

(a) Here $k(x) = \cos(\sqrt{x})$, so in this case

$$k(x) = g(f(x)) = g(u),$$

where

$$g(u) = \cos u \text{ and } u = f(x) = \sqrt{x} = x^{1/2}.$$

Since $g'(u) = -\sin u$ and $f'(x) = \frac{1}{2}x^{-1/2}$, we have

$$\begin{aligned} k'(x) &= -\sin u \times \tfrac{1}{2}x^{-1/2} \\ &= -\sin(\sqrt{x}) \times \tfrac{1}{2}x^{-1/2} \\ &= -\frac{1}{2\sqrt{x}}\sin(\sqrt{x}). \end{aligned}$$

(b) Here $k(x) = \sqrt{\cos x}$, so in this case

$$k(x) = g(f(x)) = g(u),$$

where

$$g(u) = u^{1/2} \text{ and } u = f(x) = \cos x.$$

Since $g'(u) = \frac{1}{2}u^{-1/2}$ and $f'(x) = -\sin x$, we have

$$\begin{aligned} k'(x) &= \tfrac{1}{2}u^{-1/2} \times (-\sin x) \\ &= -\frac{\sin x}{2\sqrt{\cos x}}. \end{aligned}$$

(c) Here $k(x) = e^{x^2/2}$, so in this case

$$k(x) = g(f(x)) = g(u),$$

where

$$g(u) = e^u \text{ and } u = f(x) = \tfrac{1}{2}x^2.$$

Since $g'(u) = e^u$ and $f'(x) = x$, we have

$$k'(x) = e^u x = xe^{x^2/2}.$$

(d) Here $k(x) = \sin(x^3)$, so in this case

$$k(x) = g(f(x)) = g(u),$$

where

$$g(u) = \sin u \text{ and } u = f(x) = x^3.$$

Since $g'(u) = \cos u$ and $f'(x) = 3x^2$, we have

$$\begin{aligned} k'(x) &= \cos u \times 3x^2 \\ &= 3x^2\cos(x^3). \end{aligned}$$

(e) Here $k(x) = \tan(\ln x)$, so in this case

$$k(x) = g(f(x)) = g(u),$$

where

$$g(u) = \tan u \text{ and } u = f(x) = \ln x.$$

Since $g'(u) = \sec^2 u$ and $f'(x) = 1/x$, we have

$$\begin{aligned} k'(x) &= \sec^2 u \times 1/x \\ &= \frac{\sec^2(\ln x)}{x}. \end{aligned}$$

Solution 2.6

(a) Here $f(t) = \cos^2(3t)\ln(5t)$, so we use the Product Rule and the Composite Rule:

$$\begin{aligned} f'(t) &= \frac{d}{dt}\left(\cos^2(3t)\right)\ln(5t) + \cos^2(3t)\frac{d}{dt}(\ln(5t)) \\ &= 2\cos(3t)(-3\sin(3t))\ln(5t) + \cos^2(3t)\frac{1}{t} \\ &= \cos(3t)\left(\frac{\cos(3t)}{t} - 6\sin(3t)\ln(5t)\right). \end{aligned}$$

(b) Here $f(x) = \dfrac{e^{2x}}{(x^2+1)^2}$, so we use the Quotient Rule and then the Composite Rule:

$$\begin{aligned} f'(x) &= \frac{(x^2+1)^2\frac{d}{dx}\left((e^{2x})\right) - e^{2x}\frac{d}{dx}\left((x^2+1)^2\right)}{(x^2+1)^4} \\ &= \frac{(x^2+1)^2 \times 2e^{2x} - e^{2x} \times 2(x^2+1)2x}{(x^2+1)^4} \\ &= \frac{2e^{2x}(x^2+1)(x^2+1-2x)}{(x^2+1)^4} \\ &= \frac{2e^{2x}(x-1)^2}{(x^2+1)^3}. \end{aligned}$$

(c) Here $f(x) = \sin((x^2+4)e^{3x})$, so we use the Composite Rule and then the Product Rule:

$$\begin{aligned} f'(x) &= \cos\left((x^2+4)e^{3x}\right)\frac{d}{dx}\left((x^2+4)e^{3x}\right) \\ &= \cos\left((x^2+4)e^{3x}\right)\left(2xe^{3x} + (x^2+4)3e^{3x}\right) \\ &= e^{3x}\left(3x^2+2x+12\right)\cos\left((x^2+4)e^{3x}\right). \end{aligned}$$

(d) Here $f(t) = e^{t^3/\sin t}$, so we use the Composite Rule and then the Quotient Rule:

$$\begin{aligned} f'(t) &= e^{t^3/\sin t}\frac{d}{dt}\left(\frac{t^3}{\sin t}\right) \\ &= e^{t^3/\sin t}\left(\frac{\sin t \times 3t^2 - t^3 \times \cos t}{\sin^2 t}\right) \\ &= \left(\frac{3\sin t - t\cos t}{\sin^2 t}\right)t^2 e^{t^3/\sin t}. \end{aligned}$$

Solution 2.7

(a) If $y = f(x) = \arcsin(2x)$, where $-\frac{1}{2} < x < \frac{1}{2}$, then

$$x = \tfrac{1}{2}\sin y, \text{ where } -\tfrac{1}{2}\pi < y < \tfrac{1}{2}\pi.$$

So $\dfrac{dx}{dy} = \frac{1}{2}\cos y$.

Thus, by the Inverse Rule,

$$\frac{dy}{dx} = 1\Big/\frac{dx}{dy} = \frac{2}{\cos y}, \text{ provided } \cos y \neq 0.$$

Now $\cos^2 y + \sin^2 y = 1$, so

$$\cos y = \pm\sqrt{1-\sin^2 y} = \pm\sqrt{1-(2x)^2}.$$

Since $\cos y$ is never negative in the interval $-\pi/2 < y < \pi/2$ we take the positive root:

$$\cos y = \sqrt{1-4x^2}.$$

Hence

$$\frac{dy}{dx} = \frac{2}{\sqrt{1-4x^2}}.$$

(b) (i) If $y = \arctan(3x)$, then $x = \frac{1}{3}\tan y$, so

$$\begin{aligned} \frac{dx}{dy} &= \tfrac{1}{3}\sec^2 y = \tfrac{1}{3}\left(1+\tan^2 y\right) \\ &= \tfrac{1}{3}\left(1+(3x)^2\right) = \tfrac{1}{3}\left(1+9x^2\right). \end{aligned}$$

Thus, by the Inverse Rule,

$$\frac{dy}{dx} = 1\Big/\frac{dx}{dy} = \frac{3}{1+9x^2}.$$

(ii) If $y = \arctan u$, where $u = 3x$, then

$$\frac{dy}{du} = \frac{1}{1+u^2} \text{ and } \frac{du}{dx} = 3.$$

So, by the Chain Rule,

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{1+u^2} \times 3 \\ &= \frac{3}{1+(3x)^2} \\ &= \frac{3}{1+9x^2}, \end{aligned}$$

confirming the answer in part (i).

Solution 3.1

(a) $f(x) = \dfrac{3x-8}{x-4}$.

Step 1: The denominator $x-4$ is zero when $x = 4$, so the domain of f is $\mathbb{R}$ except for 4.

Step 2: f is neither odd nor even.

Step 3: The only x-intercept occurs when $3x - 8 = 0$; that is, when $x = \frac{8}{3}$. The y-intercept is $f(0) = 2$.

Step 4: Here is a sign table for $f(x)$.

	$(-\infty,\frac{8}{3})$	$\frac{8}{3}$	$(\frac{8}{3},4)$	4	$(4,\infty)$
$3x-8$	$-$	0	$+$	$+$	$+$
$x-4$	$-$	$-$	$-$	0	$+$
$f(x)$	$+$	0	$-$	$*$	$+$

Thus f is positive on $(-\infty,\frac{8}{3})$ and $(4,\infty)$, and negative on $(\frac{8}{3},4)$.

Step 5: The derivative of f is

$$f'(x) = \frac{(x-4)\times 3-(3x-8)\times 1}{(x-4)^2}$$

$$= \frac{-4}{(x-4)^2}.$$

So $f'(x)<0$ for all x in the domain. Thus f is decreasing on $(-\infty,4)$ and $(4,\infty)$.

Step 6: Since the denominator is 0 when $x=4$, the line $x=4$ is a vertical asymptote. Also

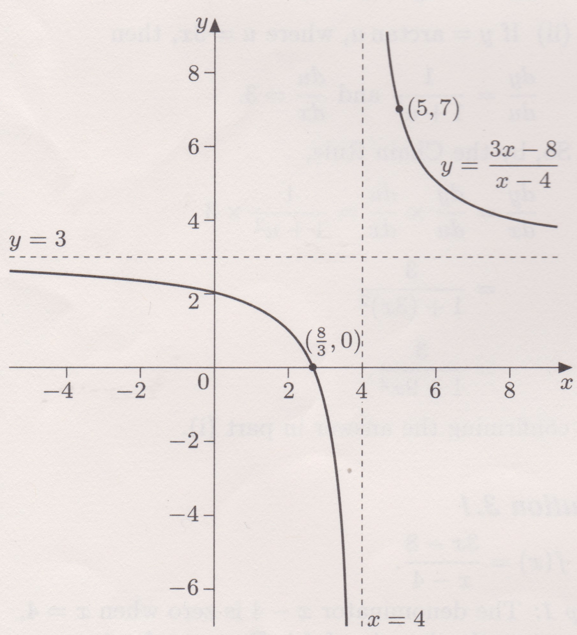
$$f(x) = \frac{3x-8}{x-4} = \frac{3-8/x}{1-4/x}$$

$$\to \frac{3}{1} = 3 \text{ as } x\to\pm\infty.$$

So the line $y=3$ is a horizontal asymptote.

To locate the curve in the interval $(4,\infty)$, we calculate $f(5)=7$.

Thus we can sketch the following graph.



(b) $k(x) = \dfrac{x^2+1}{x^2-1}$

Step 1: The denominator is 0 when $x=\pm1$, so the domain of k is $\mathbb{R}$ except for ±1.

Step 2: We have

$$k(-x) = \frac{(-x)^2+1}{(-x)^2-1} = \frac{x^2+1}{x^2-1},$$

so k is an even function.

Step 3: The x-intercept would occur when $x^2+1=0$, which has no real solutions, so there is no x-intercept. The y-intercept is $k(0)=-1$.

Step 4: Here is a sign table for $k(x)$.

	$(-\infty,-1)$	-1	$(-1,1)$	1	$(1,\infty)$
x^2+1	$+$	$+$	$+$	$+$	$+$
x^2-1	$+$	0	$-$	0	$+$
$k(x)$	$+$	$*$	$-$	$*$	$+$

Thus k is positive on $(-\infty,-1)$ and $(1,\infty)$, and negative on $(-1,1)$.

Step 5: The derivative of k is

$$k'(x) = \frac{(x^2-1)2x-(x^2+1)2x}{(x^2-1)^2}$$

$$= \frac{-4x}{(x^2-1)^2}.$$

Here is a sign table for $k'(x)$.

	$(-\infty,-1)$	-1	$(-1,0)$	0	$(0,1)$	1	$(1,\infty)$
$-4x$	$+$	$+$	$+$	0	$-$	$-$	$-$
$(x^2-1)^2$	$+$	0	$+$	$+$	$+$	0	$+$
$k'(x)$	$+$	$*$	$+$	0	$-$	$*$	$-$

So k is increasing on $(-\infty,-1)$ and $(-1,0)$, and decreasing on $(0,1)$ and $(1,\infty)$. Also k has a stationary point at $x=0$, which is a maximum, with value $k(0)=-1$.

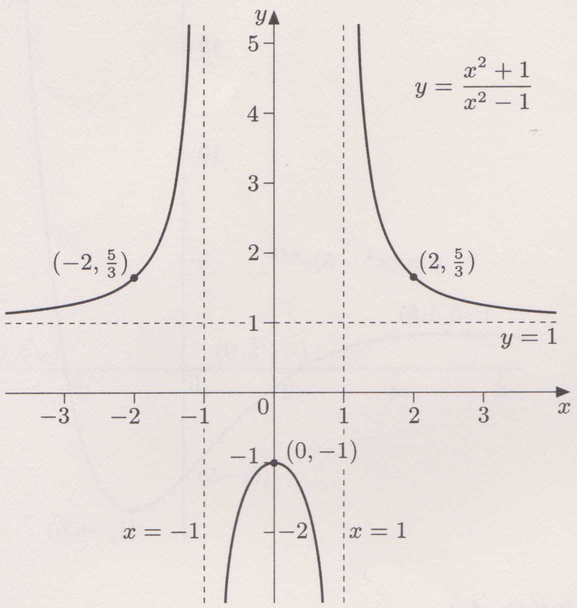
Step 6: Since the denominator is 0 when $x=\pm1$ the lines $x=1$ and $x=-1$ are vertical asymptotes. Also

$$k(x) = \frac{1+1/x^2}{1-1/x^2} \to 1 \text{ as } x\to\pm\infty,$$

so $y=1$ is a horizontal asymptote.

To locate the curve in the interval $(-\infty,-1)$, we calculate $k(-2)=5/3$ and to locate the curve in the interval $(1,\infty)$ we calculate $k(2)=5/3$.

Thus we can sketch the following graph.



(c) $h(x) = x^3 - 9x = x(x^2 - 9) = x(x-3)(x+3)$.

Step 1: The domain of h is $\mathbb{R}$.

Step 2: We have

$$h(-x) = (-x)^3 - 9(-x)$$
$$= -x^3 + 9x = -(x^3 - 9x),$$

so h is an odd function.

Step 3: The x-intercepts occur when $x^3 - 9x = 0$; that is, when $x(x-3)(x+3) = 0$. So the x-intercepts are -3, 0 and 3. Since $h(0) = 0$, the only y-intercept is zero; that is, the graph passes through the origin.

Step 4: Here is a sign table for $h(x)$.

	$(-\infty, -3)$	-3	$(-3, 0)$	0	$(0, 3)$	3	$(3, \infty)$
x	$-$	$-$	$-$	0	$+$	$+$	$+$
$x-3$	$-$	$-$	$-$	$-$	$-$	0	$+$
$x+3$	$-$	0	$+$	$+$	$+$	$+$	$+$
$x^3 - 9x$	$-$	0	$+$	0	$-$	0	$+$

Thus h is negative on $(-\infty, -3)$ and $(0, 3)$, and positive on $(-3, 0)$ and $(3, \infty)$.

Step 5: The derivative of h is

$$h'(x) = 3x^2 - 9$$
$$= 3(x^2 - 3)$$
$$= 3(x - \sqrt{3})(x + \sqrt{3}).$$

Here is a sign table for $h'(x)$.

	$(-\infty, -\sqrt{3})$	$-\sqrt{3}$	$(-\sqrt{3}, \sqrt{3})$	$\sqrt{3}$	$(\sqrt{3}, \infty)$
$x - \sqrt{3}$	$-$	$-$	$-$	0	$+$
$x + \sqrt{3}$	$-$	0	$+$	$+$	$+$
$h'(x)$	$+$	0	$-$	0	$+$

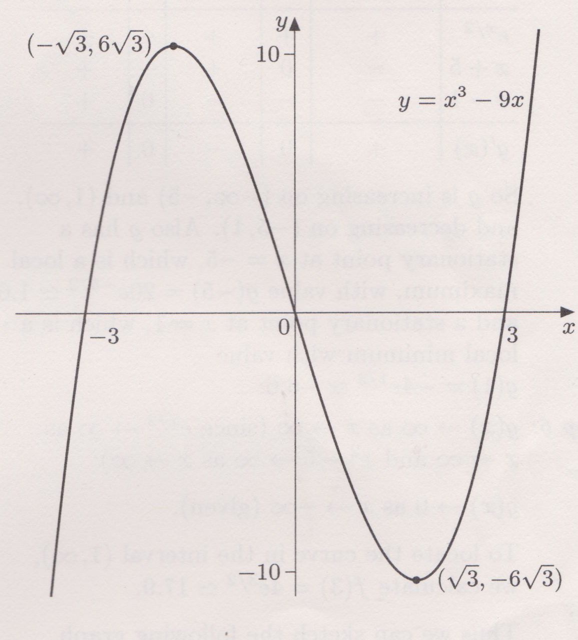
So h is increasing on $(-\infty, -\sqrt{3})$ and $(\sqrt{3}, \infty)$, and decreasing on $(-\sqrt{3}, \sqrt{3})$. Also h has a stationary point at $x = -\sqrt{3} \simeq -1.7$, which is a local maximum with value $h(-\sqrt{3}) = 6\sqrt{3} \simeq 10.4$, and a stationary point at $\sqrt{3} \simeq 1.7$, which is a local minimum with value $h(\sqrt{3}) = -6\sqrt{3} \simeq -10.4$.

Step 6: $\quad f(x) \to \infty$ as $x \to \infty$

and

$f(x) \to -\infty$ as $x \to -\infty$

Thus we can sketch the following graph.



(d) $g(x) = (x^2 - 5)e^{x/2}$

Step 1: The domain of g is $\mathbb{R}$.

Step 2: We have

$$g(-x) = ((-x)^2 - 5)e^{-x/2}$$
$$= (x^2 - 5)e^{-x/2},$$

so g is neither odd nor even.

Step 3: The x-intercepts occur when $x = \pm\sqrt{5}$. The y-intercept is $g(0) = -5$.

Step 4: Here is a sign table for $g(x)$.

	$(-\infty, -\sqrt{5})$	$-\sqrt{5}$	$(-\sqrt{5}, \sqrt{5})$	$\sqrt{5}$	$(\sqrt{5}, \infty)$
$x^2 - 5$	$+$	0	$-$	0	$+$
$e^{x/2}$	$+$	$+$	$+$	$+$	$+$
$g(x)$	$+$	0	$-$	0	$+$

So g is positive on $(-\infty, -\sqrt{5})$ and $(\sqrt{5}, \infty)$, and negative on $(-\sqrt{5}, \sqrt{5})$.

Step 5: The derivative of g is

$$\begin{aligned} g'(x) &= 2xe^{x/2} + (x^2 - 5)\tfrac{1}{2}e^{x/2} \\ &= \tfrac{1}{2}e^{x/2}(x^2 + 4x - 5) \\ &= \tfrac{1}{2}e^{x/2}(x + 5)(x - 1). \end{aligned}$$

So $g'(x) = 0$ when $x = -5$ and $x = 1$.

Here is a sign table for $g'(x)$.

	$(-\infty, -5)$	-5	$(-5, 1)$	1	$(1, \infty)$
$e^{x/2}$	$+$	$+$	$+$	$+$	$+$
$x + 5$	$-$	0	$+$	$+$	$+$
$x - 1$	$-$	$-$	$-$	0	$+$
$g'(x)$	$+$	0	$-$	0	$+$

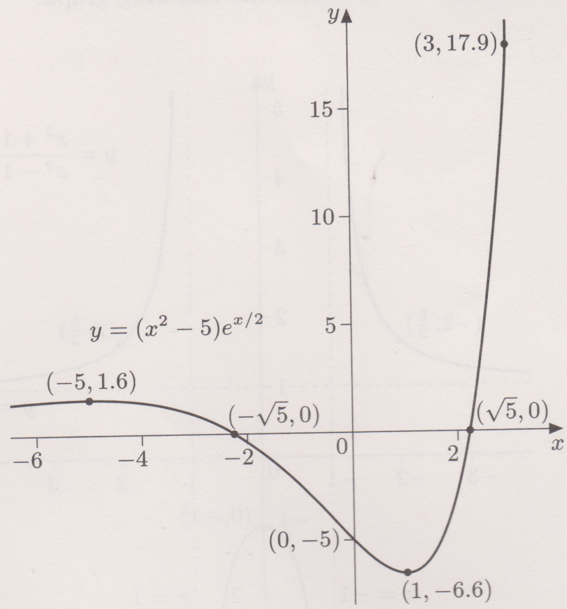
So g is increasing on $(-\infty, -5)$ and $(1, \infty)$, and decreasing on $(-5, 1)$. Also g has a stationary point at $x = -5$, which is a local maximum, with value $g(-5) = 20e^{-5/2} \simeq 1.6$ and a stationary point at $x = 1$, which is a local minimum with value $g(1) = -4e^{1/2} \simeq -6.6$.

Step 6: $g(x) \to \infty$ as $x \to \infty$ (since $e^{x/2} \to \infty$ as $x \to \infty$ and $x^2 - 5 \to \infty$ as $x \to \infty$)

$g(x) \to 0$ as $x \to -\infty$ (given).

To locate the curve in the interval $(1, \infty)$, we calculate $f(3) = 4e^{3/2} \simeq 17.9$.

Thus we can sketch the following graph.



Solution 4.1

(a) (i) With $f(x) = x^3 + 3x^2 - 2$ we have $f(-3) = -27 + 27 - 2 < 0$ and $f(-2) = -8 + 12 - 2 > 0$. Thus the function f changes sign in the interval $(-3, -2)$, so it has a zero in that interval.

(ii) Since $f'(x) = 3x^2 + 6x$, the Newton–Raphson formula is

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - \frac{x_n^3 + 3x_n^2 - 2}{3x_n^2 + 6x_n} \\ &= \frac{3x_n^3 + 6x_n^2 - (x_n^3 + 3x_n^2 - 2)}{3x_n^2 + 6x_n} \\ &= \frac{2x_n^3 + 3x_n^2 + 2}{3x_n(x_n + 2)} \qquad (n = 0, 1, 2, \ldots). \end{aligned}$$

(iii) To full calculator accuracy

$$\begin{aligned} x_0 &= -2.5, \\ x_1 &= -2.8, \\ x_2 &= -2.735\,714\,286, \\ x_3 &= -2.732\,062\,373, \\ x_4 &= -2.732\,050\,808. \end{aligned}$$

(iv) We obtain with a calculator

$$f(x_4) = 0 \text{ (to 8 d.p.)}.$$

Thus x_4 is indeed a good approximation to a solution of $f(x) = 0$.

(b) (i) With $f(x) = e^x - 3x$ we have $f(0) = 1 > 0$ and $f(1) = e - 3 < 0$. Thus the function f changes sign in the interval $(0, 1)$, so it has a zero in that interval.

(ii) Since $f'(x) = e^x - 3$, the Newton–Raphson formula is

$$\begin{aligned}
x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\
&= x_n - \frac{e^{x_n} - 3x_n}{e^{x_n} - 3} \\
&= \frac{x_n(e^{x_n} - 3) - (e^{x_n} - 3x_n)}{e^{x_n} - 3} \\
&= \frac{x_n e^{x_n} - 3x_n - e^{x_n} + 3x_n}{e^{x_n} - 3} \\
&= \frac{e^{x_n}(x_n - 1)}{e^{x_n} - 3} \quad (n = 0, 1, 2, \ldots).
\end{aligned}$$

(iii) To full calculator accuracy

$$\begin{aligned}
x_0 &= 0.5, \\
x_1 &= 0.610\,059\,655, \\
x_2 &= 0.618\,996\,7797, \\
x_3 &= 0.619\,061\,2834, \\
x_4 &= 0.619\,061\,2867.
\end{aligned}$$

(iv) We obtain with a calculator

$f(x_4) = 0$ (to 10 d.p.).

Thus x_4 is indeed a good approximation to a solution of $f(x) = 0$.

Solution 4.2

(a) (i) With $f(x) = x^4 - 3x - 3$ we have $f(1) = -5 < 0$ and $f(2) = 7 > 0$. Thus the function f changes sign in the interval $(1, 2)$, so it has a zero in that interval.

(ii) Since $f'(x) = 4x^3 - 3$, the Newton–Raphson formula is

$$\begin{aligned}
x_{n+1} &= x_n - \frac{x_n^4 - 3x_n - 3}{4x_n^3 - 3} \\
&= \frac{x_n(4x_n^3 - 3) - (x_n^4 - 3x_n - 3)}{4x_n^3 - 3} \\
&= \frac{4x_n^4 - 3x_n - x_n^4 + 3x_n + 3}{4x_n^3 - 3} \\
&= \frac{3x_n^4 + 3}{4x_n^3 - 3} \quad (n = 0, 1, 2, \ldots).
\end{aligned}$$

(iii) To full calculator accuracy

$$\begin{aligned}
x_0 &= 1.5, \\
x_1 &= 1.732\,142\,857, \\
x_2 &= 1.686\,860\,180, \\
x_3 &= 1.684\,621\,010, \\
x_4 &= 1.684\,615\,706.
\end{aligned}$$

(iv) We obtain with a calculator $f(x_4) = 0$ (to 8 d.p.).

Thus x_4 is indeed a good approximation to a solution of $f(x) = 0$.

(v) The Newton–Raphson method fails in this case if we choose x_0 to be the stationary point of f, which is $\sqrt[3]{3/4} \simeq 0.91$.

(b) (i) With $f(x) = \frac{1}{2}x + \cos x$ we have $f(-2) \simeq -1.42 < 0$ and $f(-1) \simeq 0.04 > 0$. Thus the function f changes sign in the interval $(-2, -1)$, so it has a zero in that interval.

(ii) Since $f'(x) = \frac{1}{2} - \sin x$, the Newton–Raphson formula is

$$\begin{aligned}
x_{n+1} &= x_n - \frac{\frac{1}{2}x_n + \cos x_n}{\frac{1}{2} - \sin x_n} \\
&= \frac{x_n(\frac{1}{2} - \sin x_n) - (\frac{1}{2}x_n + \cos x_n)}{\frac{1}{2} - \sin x_n} \\
&= \frac{-x_n \sin x_n - \cos x_n}{\frac{1}{2} - \sin x_n} \\
&= \frac{x_n \sin x_n + \cos x_n}{\sin x_n - \frac{1}{2}} \quad (n = 0, 1, 2, \ldots).
\end{aligned}$$

(iii) To full calculator accuracy

$$\begin{aligned}
x_0 &= -1.5, \\
x_1 &= -1.046\,400\,619, \\
x_2 &= -1.029\,917\,121, \\
x_3 &= -1.029\,866\,530, \\
x_4 &= -1.029\,866\,529.
\end{aligned}$$

(iv) We obtain with a calculator $f(x_4) = 0$ (to 9 d.p.).

Thus x_4 is indeed a good approximation to a solution of $f(x) = 0$.

(v) The Newton–Raphson method fails in this case if we take $x_0 = \frac{1}{6}\pi$, since this is a stationary point of f.

(vi) The Newton–Raphson method fails if we take x_0 to be any stationary point of f. These stationary points are the solutions of

$$\sin x - \tfrac{1}{2} = 0;$$

that is, all numbers of the form

$$x = 2n\pi + \tfrac{1}{6}\pi \quad \text{or} \quad x = 2n\pi + \tfrac{5}{6}\pi,$$

where $n \in \mathbb{Z}$.

Solutions for Chapter C2

Solution 1.1

In each case c is an arbitrary constant.

(a) $\displaystyle\int (3x^2 + \cos(4x))\,dx = x^3 + \tfrac{1}{4}\sin(4x) + c$

(b) $\displaystyle\int (e^{2t} - \sin(5t))\,dt = \tfrac{1}{2}e^{2t} + \tfrac{1}{5}\cos(5t) + c$

(c) $\displaystyle\int \frac{4}{1+t^2}\,dt = 4\arctan t + c$

(d) $\displaystyle\int \left(\frac{e^{3x}+e^{2x}}{e^x}\right)dx = \int\left(\frac{e^{3x}}{e^x} + \frac{e^{2x}}{e^x}\right)dx$

$$= \int (e^{2x} + e^x)\,dx = \tfrac{1}{2}e^{2x} + e^x + c$$

Solution 1.2

(a)
$$\begin{aligned}\int_1^2 (3x^4 - e^{2x})\,dx &= \left[\tfrac{3}{5}x^5 - \tfrac{1}{2}e^{2x}\right]_1^2\\
&= \left(\tfrac{3}{5}\times 32 - \tfrac{1}{2}\times e^4\right) - \left(\tfrac{3}{5} - \tfrac{1}{2}e^2\right)\\
&= \tfrac{3}{5}\times 31 + \tfrac{1}{2}(e^2 - e^4)\\
&= -5.0045 \text{ (to 4 d.p.)}\end{aligned}$$

(b)
$$\begin{aligned}\int_{-\pi}^{\pi} \cos\left(\tfrac{1}{5}t\right)dt &= \left[5\sin\left(\tfrac{1}{5}t\right)\right]_{-\pi}^{\pi}\\
&= 5\left(\sin(\pi/5) - \sin(-\pi/5)\right)\\
&= 10\sin(\pi/5)\\
&= 5.8779 \text{ (to 4 d.p.)}\end{aligned}$$

(c)
$$\begin{aligned}\int_{-1/2}^{1/2} \frac{1}{\sqrt{1-u^2}}\,du &= \left[\arcsin u\right]_{-1/2}^{1/2}\\
&= \arcsin(\tfrac{1}{2}) - \arcsin\left(-\tfrac{1}{2}\right)\\
&= \pi/6 - (-\pi/6)\\
&= \pi/3\\
&\simeq 1.0472 \text{ (to 4 d.p.)}\end{aligned}$$

(d)
$$\begin{aligned}\int_0^1 e^{t/2}\,dt &= \left[2e^{t/2}\right]_0^1\\
&= 2e^{1/2} - 2\\
&= 1.2974 \text{ (to 4 d.p.)}\end{aligned}$$

Solution 1.3

(a) (i) For x in the interval $[-3,-2]$, x is negative, but any negative number to an even power is positive, so x^{10} is positive. Hence the graph of $f(x) = x^{10}$ is above the x-axis on the interval $[-3,-2]$.

(ii) The area is

$$\begin{aligned}\int_{-3}^{-2} x^{10}\,dx &= \left[\tfrac{1}{11}x^{11}\right]_{-3}^{-2}\\
&= \tfrac{1}{11}\left((-2)^{11} - (-3)^{11}\right)\\
&= 15\,918 \text{ (to nearest integer).}\end{aligned}$$

(b) (i) For x in the interval $[\pi/15, 2\pi/15]$, $5x$ lies in the interval $[\pi/3, 2\pi/3]$ so $\sin(5x)$ is positive. Hence the graph of $y = \sin(5x)$ is above the x-axis on the interval $[\pi/15, 2\pi/15]$.

(ii) The area is

$$\begin{aligned}\int_{\pi/15}^{2\pi/15} \sin(5x)\,dx &= \left[-\tfrac{1}{5}\cos(5x)\right]_{\pi/15}^{2\pi/15}\\
&= -\tfrac{1}{5}\left(\cos(2\pi/3) - \cos(\pi/3)\right)\\
&= -\tfrac{1}{5}\left(-\tfrac{1}{2} - \tfrac{1}{2}\right)\\
&= \tfrac{1}{5} = 0.2.\end{aligned}$$

(c) (i)
$$\begin{aligned}\int_2^3 (e^{x/2} - 1)\,dx &= \left[2e^{x/2} - x\right]_2^3\\
&= (2e^{3/2} - 3) - (2e - 2)\\
&= 2.5268 \text{ (to 4 d.p.)}\end{aligned}$$

$$\begin{aligned}\int_{-1}^2 (e^{x/2} - 1)\,dx &= \left[2e^{x/2} - x\right]_{-1}^2\\
&= (2e - 2) - (2e^{-1/2} + 1)\\
&= 1.2235 \text{ (to 4 d.p.)}\end{aligned}$$

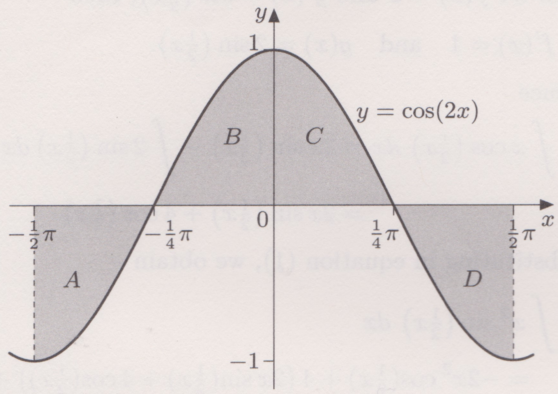
(ii) The graph of $y = e^{x/2} - 1$ passes through the origin, and $e^{x/2} - 1$ is positive for $x > 0$ and negative for $x < 0$. Hence only $\displaystyle\int_2^3 (e^{x/2} - 1)\,dx$ represents the area bounded by the graph, the x-axis and the given limits for x.

(iii) To find the required area it is necessary to find the (positive) area bounded by $x = 0$ and $x = 3$, which is above the x-axis, and then subtract from it the (negative) area bounded by $x = -1$ and $x = 0$, which is below the x-axis.

The required area is

$$\int_0^3 (e^{x/2}-1)\,dx - \int_{-1}^0 (e^{x/2}-1)\,dx$$
$$= \left[2e^{x/2}-x\right]_0^3 - \left[2e^{x/2}-x\right]_{-1}^0$$
$$= (2e^{3/2}-3) - 2 - (2-(2e^{-1/2}+1))$$
$$= 2e^{3/2} + 2e^{-1/2} - 6$$
$$= 4.1764 \text{ (to 4 d.p.)}.$$

(d) The graph of $y = \cos(2x)$ is shown below.

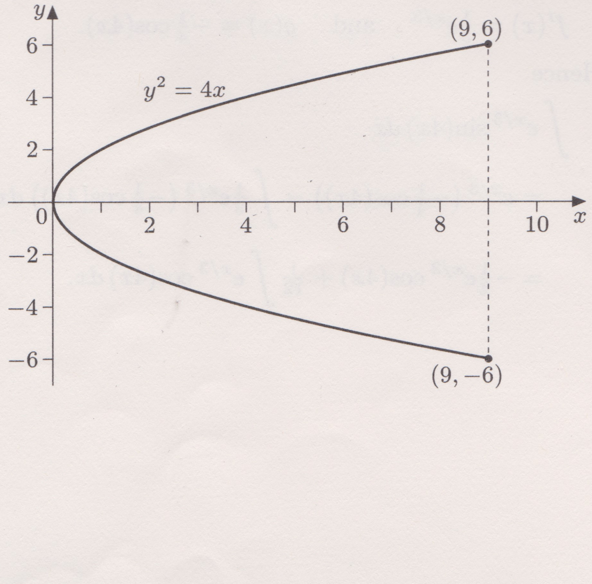


By properties of the cosine function, the area bounded by the curve and the x-axis, between $x = -\pi/2$ and $x = \pi/2$ is made up of four equal parts A, B, C and D. So the total area equals 4 times the area from $x = 0$ to $x = \pi/4$. Hence the required area is

$$4\int_0^{\pi/4} \cos(2x)\,dx = 4\left[\tfrac{1}{2}\sin(2x)\right]_0^{\pi/4}$$
$$= 2(\sin(\pi/2) - \sin 0)$$
$$= 2,$$

as required.

(e) The curve $y^2 = 4x$ is a parabola, as shown below.



The parabola is symmetrical about the x-axis, so the area required is twice the area bounded by that part of the curve above the x-axis from $x = 0$ to $x = 9$ and the x-axis.

The part of the curve above the x-axis has equation $y = 2\sqrt{x}$, since $y \geq 0$ on this part.

Hence the total area is

$$2\int_0^9 2x^{1/2}\,dx = 4\left[\tfrac{2}{3}x^{3/2}\right]_0^9$$
$$= \tfrac{8}{3}(9^{3/2} - 0)$$
$$= \frac{8\times 27}{3} = 72.$$

Solution 2.1

In each case we use the integration by parts formula

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx,$$

and c is an arbitrary constant.

(a) Let $f(x) = x$ and $g'(x) = \cos(3x)$; then

$$f'(x) = 1 \quad\text{and}\quad g(x) = \tfrac{1}{3}\sin(3x).$$

Hence

$$\int x\cos(3x)\,dx = \tfrac{1}{3}x\sin(3x) - \int 1\times\tfrac{1}{3}\sin(3x)\,dx$$
$$= \tfrac{1}{3}x\sin(3x) - \tfrac{1}{3}\int \sin(3x)\,dx$$
$$= \tfrac{1}{3}x\sin(3x) + \tfrac{1}{9}\cos(3x) + c.$$

(b) Let $f(x) = 2x$ and $g'(x) = \sin\left(\tfrac{1}{5}x\right)$; then

$$f'(x) = 2 \text{ and } g(x) = -5\cos(\tfrac{1}{5}x).$$

Hence

$$\int 2x\sin\left(\tfrac{1}{5}x\right)\,dx$$
$$= 2x\left(-5\cos\left(\tfrac{1}{5}x\right)\right) - \int 2\left(-5\cos\left(\tfrac{1}{5}x\right)\right)\,dx$$
$$= -10x\cos\left(\tfrac{1}{5}x\right) + 10\int \cos\left(\tfrac{1}{5}x\right)\,dx$$
$$= -10x\cos\left(\tfrac{1}{5}x\right) + 50\sin\left(\tfrac{1}{5}x\right) + c.$$

(c) Let $f(x) = \ln(5x)$ and $g'(x) = x$; then

$$f'(x) = 1/x \quad\text{and}\quad g(x) = \tfrac{1}{2}x^2.$$

Hence

$$\int x\ln(5x)\,dx = \ln(5x)\times\tfrac{1}{2}x^2 - \int \frac{1}{x}\times\tfrac{1}{2}x^2\,dx$$
$$= \tfrac{1}{2}x^2\ln(5x) - \tfrac{1}{2}\int x\,dx$$
$$= \tfrac{1}{2}x^2\ln(5x) - \tfrac{1}{4}x^2 + c.$$

Solution 2.2

(a) Let $f(x) = x$ and $g'(x) = e^{4x}$; then

$$f'(x) = 1 \quad \text{and} \quad g(x) = \tfrac{1}{4}e^{4x}.$$

Hence

$$\int_0^1 xe^{4x}\,dx = \left[\tfrac{1}{4}xe^{4x}\right]_0^1 - \tfrac{1}{4}\int_0^1 e^{4x}\,dx$$

$$= \tfrac{1}{4}e^4 - \tfrac{1}{4}\left[\tfrac{1}{4}e^{4x}\right]_0^1$$

$$= \tfrac{1}{4}e^4 - \tfrac{1}{16}\left(e^4 - 1\right)$$

$$= \tfrac{3}{16}e^4 + \tfrac{1}{16}$$

$$= 10.300 \text{ (to 3 d.p.)}.$$

(b) Let $f(x) = x$ and $g'(x) = \sin(2x)$; then

$$f'(x) = 1 \quad \text{and} \quad g(x) = -\tfrac{1}{2}\cos(2x).$$

Hence

$$\int_{-\pi/4}^{\pi/4} x\sin(2x)\,dx$$

$$= \left[-\tfrac{1}{2}x\cos(2x)\right]_{-\pi/4}^{\pi/4} - \int_{-\pi/4}^{\pi/4}\left(-\tfrac{1}{2}\cos(2x)\right)dx$$

$$= -\tfrac{1}{2}\left(\pi/4\cos(\pi/2) - (-\pi/4\cos(-\pi/2))\right)$$

$$+\tfrac{1}{2}\left[\tfrac{1}{2}\sin(2x)\right]_{-\pi/4}^{\pi/4}$$

$$= \tfrac{1}{4}(\sin(\pi/2) - \sin(-\pi/2))$$

$$= \tfrac{1}{4}(1-(-1))$$

$$= 0.5.$$

(c) Let $f(x) = \ln(4x)$ and $g'(x) = x^2$; then

$$f'(x) = 1/x \quad \text{and} \quad g(x) = \tfrac{1}{3}x^3.$$

Hence

$$\int_1^2 x^2\ln(4x)\,dx = \left[\tfrac{1}{3}x^3\ln(4x)\right]_1^2 - \int_1^2 \frac{1}{x}\times\tfrac{1}{3}x^3\,dx$$

$$= \tfrac{1}{3}(8\ln 8 - \ln 4) - \tfrac{1}{3}\int_1^2 x^2\,dx$$

$$= \tfrac{1}{3}(8\ln 8 - \ln 4) - \tfrac{1}{9}\left[x^3\right]_1^2$$

$$= \tfrac{1}{3}(8\ln 8 - \ln 4) - \tfrac{1}{9}(8-1)$$

$$= 4.305 \text{ (to 3 d.p.)}.$$

Solution 2.3

(a) Let $f(x) = x^2$ and $g'(x) = \sin\left(\tfrac{1}{2}x\right)$; then

$$f'(x) = 2x \quad \text{and} \quad g(x) = -2\cos\left(\tfrac{1}{2}x\right).$$

Hence

$$\int x^2\sin\left(\tfrac{1}{2}x\right)dx$$

$$= -2x^2\cos\left(\tfrac{1}{2}x\right) - \int 2x\left(-2\cos\left(\tfrac{1}{2}x\right)\right)dx$$

$$= -2x^2\cos\left(\tfrac{1}{2}x\right) + 4\int x\cos\left(\tfrac{1}{2}x\right)dx. \qquad (1)$$

Now let $f(x) = x$ and $g'(x) = \cos\left(\tfrac{1}{2}x\right)$; then

$$f'(x) = 1 \quad \text{and} \quad g(x) = 2\sin\left(\tfrac{1}{2}x\right).$$

Hence

$$\int x\cos\left(\tfrac{1}{2}x\right)dx = 2x\sin\left(\tfrac{1}{2}x\right) - \int 2\sin\left(\tfrac{1}{2}x\right)dx$$

$$= 2x\sin\left(\tfrac{1}{2}x\right) + 4\cos\left(\tfrac{1}{2}x\right).$$

Substituting in equation (1), we obtain

$$\int x^2\sin\left(\tfrac{1}{2}x\right)dx$$

$$= -2x^2\cos(\tfrac{1}{2}x) + 4\left(2x\sin(\tfrac{1}{2}x) + 4\cos(\tfrac{1}{2}x)\right) + c$$

$$= -2x^2\cos\left(\tfrac{1}{2}x\right) + 8x\sin\left(\tfrac{1}{2}x\right)$$

$$+16\cos\left(\tfrac{1}{2}x\right) + c$$

$$= 2\cos\left(\tfrac{1}{2}x\right)\left(8 - x^2\right) + 8x\sin\left(\tfrac{1}{2}x\right) + c.$$

(b) Let $f(x) = e^{x/3}$ and $g'(x) = \cos(4x)$; then

$$f'(x) = \tfrac{1}{3}e^{x/3} \quad \text{and} \quad g(x) = \tfrac{1}{4}\sin(4x).$$

Hence

$$\int e^{x/3}\cos(4x)\,dx$$

$$= e^{x/3}\tfrac{1}{4}\sin(4x) - \int \tfrac{1}{3}e^{x/3}\tfrac{1}{4}\sin(4x)\,dx$$

$$= \tfrac{1}{4}e^{x/3}\sin(4x) - \tfrac{1}{12}\int e^{x/3}\sin(4x)\,dx. \qquad (2)$$

Now let $f(x) = e^{x/3}$ and $g'(x) = \sin(4x)$; then

$$f'(x) = \tfrac{1}{3}e^{x/3} \quad \text{and} \quad g(x) = -\tfrac{1}{4}\cos(4x).$$

Hence

$$\int e^{x/3}\sin(4x)\,dx$$

$$= e^{x/3}\left(-\tfrac{1}{4}\cos(4x)\right) - \int \tfrac{1}{3}e^{x/3}\left(-\tfrac{1}{4}\cos(4x)\right)dx$$

$$= -\tfrac{1}{4}e^{x/3}\cos(4x) + \tfrac{1}{12}\int e^{x/3}\cos(4x)\,dx.$$

Substituting in equation (2), we obtain

$$\int e^{x/3}\cos(4x)\,dx$$
$$= \tfrac{1}{4}e^{x/3}\sin(4x)$$
$$-\tfrac{1}{12}\left(-\tfrac{1}{4}e^{x/3}\cos(4x)+\tfrac{1}{12}\int e^{x/3}\cos(4x)\,dx\right)$$
$$= \tfrac{1}{4}e^{x/3}\sin(4x)+\tfrac{1}{48}e^{x/3}\cos(4x)$$
$$-\tfrac{1}{144}\int e^{x/3}\cos(4x)\,dx.$$

So

$$\left(1+\tfrac{1}{144}\right)\int e^{x/3}\cos(4x)\,dx$$
$$= \tfrac{1}{4}e^{x/3}\sin(4x)+\tfrac{1}{48}e^{x/3}\cos(4x),$$

and hence

$$\int e^{x/3}\cos(4x)\,dx$$
$$= \tfrac{144}{145}\left(\tfrac{1}{4}e^{x/3}\sin(4x)+\tfrac{1}{48}e^{x/3}\cos(4x)\right)+c$$
$$= \tfrac{3}{145}e^{x/3}(12\sin(4x)+\cos(4x))+c.$$

(c) Let $f(x)=e^{-2x}$ and $g'(x)=\sin(5x)$; then

$$f'(x)=-2e^{-2x} \quad\text{and}\quad g(x)=-\tfrac{1}{5}\cos(5x).$$

Then we have

$$\int e^{-2x}\sin(5x)\,dx$$
$$= e^{-2x}\left(-\tfrac{1}{5}\cos(5x)\right)-\int\left(-2e^{-2x}\right)\left(-\tfrac{1}{5}\cos(5x)\right)dx$$
$$= -\tfrac{1}{5}e^{-2x}\cos(5x)-\tfrac{2}{5}\int e^{-2x}\cos(5x)\,dx \qquad (3)$$

Now let $f(x)=e^{-2x}$ and $g'(x)=\cos(5x)$; then

$$f'(x)=-2e^{-2x} \quad\text{and}\quad g(x)=\tfrac{1}{5}\sin(5x).$$

Hence

$$\int e^{-2x}\cos(5x)\,dx$$
$$= e^{-2x}\left(\tfrac{1}{5}\sin(5x)\right)-\int\left(-2e^{-2x}\right)\left(\tfrac{1}{5}\sin(5x)\right)dx$$
$$= \tfrac{1}{5}e^{-2x}\sin(5x)+\tfrac{2}{5}\int e^{-2x}\sin(5x)\,dx.$$

Substituting in equation (3), we obtain

$$\int e^{-2x}\sin(5x)\,dx$$
$$= -\tfrac{1}{5}e^{-2x}\cos(5x)$$
$$-\tfrac{2}{5}\left(\tfrac{1}{5}e^{-2x}\sin(5x)+\tfrac{2}{5}\int e^{-2x}\sin(5x)\,dx\right)$$
$$= -\tfrac{1}{5}e^{-2x}\cos(5x)-\tfrac{2}{25}e^{-2x}\sin(5x)$$
$$-\tfrac{4}{25}\int e^{-2x}\sin(5x)\,dx.$$

So

$$\left(1+\tfrac{4}{25}\right)\int e^{-2x}\sin(5x)\,dx$$
$$= -\tfrac{1}{5}e^{-2x}\cos(5x)-\tfrac{2}{25}e^{-2x}\sin(5x),$$

and hence

$$\int e^{-2x}\sin(5x)\,dx$$
$$= \tfrac{25}{29}\left(-\tfrac{1}{5}e^{-2x}\cos(5x)-\tfrac{2}{25}e^{-2x}\sin(5x)\right)+c$$
$$= -\tfrac{1}{29}e^{-2x}(5\cos(5x)+2\sin(5x))+c.$$

(d) Let $f(x)=\ln x$ and $g'(x)=x^2$; then

$$f'(x)=\frac{1}{x} \quad\text{and}\quad g(x)=\tfrac{1}{3}x^3.$$

Hence

$$\int x^2\ln x\,dx = \ln x\times\tfrac{1}{3}x^3-\int\frac{1}{x}\times\tfrac{1}{3}x^3\,dx$$
$$= \tfrac{1}{3}x^3\ln x-\tfrac{1}{3}\int x^2\,dx$$
$$= \tfrac{1}{3}x^3\ln x-\tfrac{1}{9}x^3+c$$
$$= \tfrac{1}{9}x^3(3\ln x-1)+c.$$

Solution 2.4

(a) Let $f(x)=x^2$ and $g'(x)=e^{5x}$; then

$$f'(x)=2x \quad\text{and}\quad g(x)=\tfrac{1}{5}e^{5x}.$$

Hence

$$\int_0^1 x^2e^{5x}\,dx = \left[x^2\left(\tfrac{1}{5}e^{5x}\right)\right]_0^1-\int_0^1 2x\left(\tfrac{1}{5}e^{5x}\right)dx$$
$$= \tfrac{1}{5}e^5-\tfrac{2}{5}\int_0^1 xe^{5x}\,dx. \qquad (1)$$

Now let $f(x)=x$ and $g'(x)=e^{5x}$; then

$$f'(x)=1 \quad\text{and}\quad g(x)=\tfrac{1}{5}e^{5x}.$$

Hence

$$\int_0^1 xe^{5x}\,dx = \left[x\left(\tfrac{1}{5}e^{5x}\right)\right]_0^1-\int_0^1 1\left(\tfrac{1}{5}e^{5x}\right)dx$$
$$= \tfrac{1}{5}e^5-\tfrac{1}{5}\int_0^1 e^{5x}\,dx$$
$$= \tfrac{1}{5}e^5-\tfrac{1}{5}\left[\tfrac{1}{5}e^{5x}\right]_0^1$$
$$= \tfrac{1}{5}e^5-\tfrac{1}{25}e^5+\tfrac{1}{25}$$
$$= \tfrac{4}{25}e^5+\tfrac{1}{25}.$$

Substituting in equation (1), we obtain

$$\int_0^1 x^2e^{5x}\,dx = \tfrac{1}{5}e^5-\tfrac{2}{5}\left(\tfrac{4}{25}e^5+\tfrac{1}{25}\right)$$
$$= \tfrac{1}{125}\left(17e^5-2\right)$$
$$= 20.1682 \text{ (to 4 d.p.).}$$

(b) Let $f(x) = e^{x/2}$ and $g'(x) = \cos(2x)$; then

$$f'(x) = \tfrac{1}{2}e^{x/2} \quad \text{and} \quad g(x) = \tfrac{1}{2}\sin(2x).$$

Hence

$$\begin{aligned}
&\int_0^{\pi/4} e^{x/2}\cos(2x)\,dx \\
&= \left[e^{x/2}(\tfrac{1}{2}\sin(2x))\right]_0^{\pi/4} - \int_0^{\pi/4} \tfrac{1}{2}e^{x/2}(\tfrac{1}{2}\sin(2x))\,dx \\
&= \tfrac{1}{2}e^{\pi/8} - \tfrac{1}{4}\int_0^{\pi/4} e^{x/2}\sin(2x)\,dx. \qquad (2)
\end{aligned}$$

Now let $f(x) = e^{x/2}$ and $g'(x) = \sin(2x)$; then

$$f'(x) = \tfrac{1}{2}e^{x/2} \quad \text{and} \quad g(x) = -\tfrac{1}{2}\cos(2x).$$

Hence

$$\begin{aligned}
\int_0^{\pi/4} e^{x/2}\sin(2x)\,dx &= \left[e^{x/2}(-\tfrac{1}{2}\cos(2x))\right]_0^{\pi/4} \\
&\quad - \int_0^{\pi/4} \tfrac{1}{2}e^{x/2}(-\tfrac{1}{2}\cos(2x))\,dx \\
&= \tfrac{1}{2} + \tfrac{1}{4}\int_0^{\pi/4} e^{x/2}\cos(2x)\,dx.
\end{aligned}$$

Substituting in equation (2), we obtain

$$\begin{aligned}
&\int_0^{\pi/4} e^{x/2}\cos(2x)\,dx \\
&= \tfrac{1}{2}e^{\pi/8} - \tfrac{1}{4}\left(\tfrac{1}{2} + \tfrac{1}{4}\int_0^{\pi/4} e^{x/2}\cos(2x)\,dx\right) \\
&= \tfrac{1}{2}e^{\pi/8} - \tfrac{1}{8} - \tfrac{1}{16}\int_0^{\pi/4} e^{x/2}\cos(2x)\,dx.
\end{aligned}$$

So

$$\left(1 + \tfrac{1}{16}\right)\int_0^{\pi/4} e^{x/2}\cos(2x)\,dx = \tfrac{1}{2}e^{\pi/8} - \tfrac{1}{8},$$

and hence

$$\begin{aligned}
\int_0^{\pi/4} e^{x/2}\cos(2x)\,dx &= \tfrac{16}{17}\left(\tfrac{1}{2}e^{\pi/8} - \tfrac{1}{8}\right) \\
&= 0.5793 \text{ (to 4 d.p.)}.
\end{aligned}$$

(c) Let $f(x) = \ln x$ and $g'(x) = x^2$; then

$$f'(x) = 1/x \quad \text{and} \quad g(x) = -\tfrac{1}{3}x^3.$$

Then

$$\begin{aligned}
&\int_1^4 x^2 \ln x\,dx \\
&= \left[\tfrac{1}{3}x^3\ln x\right]_1^4 - \int_1^4 \frac{1}{x}\left(\tfrac{1}{3}x^3\right)dx \\
&= \left[\tfrac{1}{3}x^3\ln x\right]_1^4 - \tfrac{1}{3}\int_1^4 x^2\,dx \\
&= \left[\tfrac{1}{3}x^3\ln x\right]_1^4 - \tfrac{1}{3}\left[\tfrac{1}{3}x^3\right]_1^4 \\
&= \left[\tfrac{1}{3}x^3\ln x\right]_1^4 - \left[\tfrac{1}{9}x^3\right]_1^4 \\
&= \tfrac{1}{3}\times 4^3\ln 4 - \tfrac{1}{3}\ln 1 - \tfrac{1}{9}(4^3 - 1) \\
&= \tfrac{64}{3}\ln 4 - 7 \\
&= 22.5743 \text{ (to 4 d.p.)}.
\end{aligned}$$

Solution 3.1

In each case c is an arbitrary constant.

(a) Take $u = x^5$ so $\dfrac{du}{dx} = 5x^4$. Hence

$$\begin{aligned}
\int x^4 e^{x^5}\,dx &= \tfrac{1}{5}\int e^{x^5}(5x^4)\,dx \\
&= \tfrac{1}{5}\int e^u\,du \\
&= \tfrac{1}{5}e^u + c \\
&= \tfrac{1}{5}e^{x^5} + c.
\end{aligned}$$

(b) Take $u = x^3$ so $\dfrac{du}{dx} = 3x^2$. Hence

$$\begin{aligned}
\int x^2\sin(x^3)\,dx &= \tfrac{1}{3}\int \sin(x^3)3x^2\,dx \\
&= \tfrac{1}{3}\int \sin u\,du \\
&= -\tfrac{1}{3}\cos u + c \\
&= -\tfrac{1}{3}\cos(x^3) + c.
\end{aligned}$$

(c) Take $u = 1 - 2x^3$ so $\dfrac{du}{dx} = -6x^2$. Hence

$$\begin{aligned}
\int x^2(1-2x^3)^9\,dx &= -\tfrac{1}{6}\int (1-2x^3)^9(-6x^2)\,dx \\
&= -\tfrac{1}{6}\int u^9\,du \\
&= -\tfrac{1}{6}\times\tfrac{1}{10}u^{10} + c \\
&= -\tfrac{1}{60}(1-2x^3)^{10} + c.
\end{aligned}$$

(d) Take $u = 2 - x^6$ so $\dfrac{du}{dx} = -6x^5$. Hence

$$\begin{aligned}\int \frac{x^5}{2-x^6}\,dx &= -\tfrac{1}{6}\int \frac{-6x^5}{2-x^6}\,dx\\ &= -\tfrac{1}{6}\int \frac{1}{u}\,du\\ &= -\tfrac{1}{6}\ln|u| + c\\ &= -\tfrac{1}{6}\ln|2-x^6| + c.\end{aligned}$$

Solution 3.2

(a) Take $u = 2x^2 + 1$ so $\dfrac{du}{dx} = 4x$. Also $u = 1$ when $x = 0$, and $u = 3$ when $x = 1$. Hence

$$\begin{aligned}\int_0^1 x(2x^2+1)^{1/2}\,dx &= \tfrac{1}{4}\int_0^1 (2x^2+1)^{1/2}4x\,dx\\ &= \tfrac{1}{4}\int_1^3 u^{1/2}\,du\\ &= \tfrac{1}{4}\Big[\tfrac{2}{3}u^{3/2}\Big]_1^3\\ &= \tfrac{1}{6}(3^{3/2} - 1^{3/2})\\ &= 0.6994 \text{ (to 4 d.p.)}.\end{aligned}$$

(b) Take $u = x^2 + 3$ so $\dfrac{du}{dx} = 2x$. Also $u = 3$ when $x = 0$, and $u = 4$ when $x = 1$. Hence

$$\begin{aligned}\int_0^1 xe^{x^2+3}\,dx &= \tfrac{1}{2}\int_0^1 e^{x^2+3}(2x)\,dx\\ &= \tfrac{1}{2}\int_3^4 e^u\,du\\ &= \tfrac{1}{2}\Big[e^u\Big]_3^4\\ &= \tfrac{1}{2}\left(e^4 - e^3\right)\\ &= 17.2563 \text{ (to 4 d.p.)}.\end{aligned}$$

(c) Take $u = \arctan(3x^2)$ so

$$\frac{du}{dx} = \frac{1}{1+(3x^2)^2}\times 6x = \frac{6x}{1+9x^4}.$$

Also $u = 0$ when $x = 0$, and $u = \arctan(1) = \pi/4$ when $x = 1/\sqrt{3}$. Hence

$$\begin{aligned}\int_0^{1/\sqrt{3}} \frac{x}{1+9x^4}\,dx &= \tfrac{1}{6}\int_0^{1/\sqrt{3}} \frac{6x}{1+9x^4}\,dx\\ &= \tfrac{1}{6}\int_0^{\pi/4} 1\,du\\ &= \tfrac{1}{6}\Big[u\Big]_0^{\pi/4}\\ &= \tfrac{1}{6}(\pi/4 - 0)\\ &= \pi/24 = 0.1309 \text{ (to 4 d.p.)}.\end{aligned}$$

(d) Take $u = 1 + \sin^2 x$ so $\dfrac{du}{dx} = 2\sin x\cos x$.

Also $u = 1$ when $x = 0$, and $u = 2$ when $x = \pi/2$. Hence

$$\begin{aligned}\int_0^{\pi/2} \frac{\cos x\sin x}{1+\sin^2 x}\,dx &= \tfrac{1}{2}\int_0^{\pi/2}\frac{2\cos x\sin x}{1+\sin^2 x}\,dx\\ &= \tfrac{1}{2}\int_1^2 \frac{du}{u}\\ &= \tfrac{1}{2}\Big[\ln|u|\Big]_1^2\\ &= \tfrac{1}{2}\ln 2\\ &= 0.3466 \text{ (to 4 d.p.)}.\end{aligned}$$

Solution 3.3

In each case c is an arbitrary constant.

(a) Let $x = \tfrac{1}{3}(u+1)$, where $u = 3x - 1$. Then $\dfrac{dx}{du} = \tfrac{1}{3}$. Hence

$$\begin{aligned}\int \frac{x}{(3x-1)^4}\,dx &= \tfrac{1}{3}\int \tfrac{1}{3}\frac{(u+1)}{u^4}\,du\\ &= \tfrac{1}{9}\int (u^{-3} + u^{-4})\,du\\ &= \tfrac{1}{9}(-\tfrac{1}{2}u^{-2} - \tfrac{1}{3}u^{-3}) + c\\ &= -\tfrac{1}{54}\frac{3u+2}{u^3} + c\\ &= -\tfrac{1}{54}\frac{(9x-1)}{(3x-1)^3} + c.\end{aligned}$$

(b) Let $x = u^2 - 1$, where $u = (x+1)^{1/2}$. Then $\dfrac{dx}{du} = 2u$. Hence

$$\begin{aligned}\int \frac{x^2}{(x+1)^{1/2}}\,dx &= \int \frac{(u^2-1)^2}{u}2u\,du\\ &= \int (2u^4 - 4u^2 + 2)\,du\\ &= \tfrac{2}{5}u^5 - \tfrac{4}{3}u^3 + 2u + c\\ &= \tfrac{2}{5}(x+1)^{5/2} - \tfrac{4}{3}(x+1)^{3/2}\\ &\quad + 2(x+1)^{1/2} + c.\end{aligned}$$

(c) Let $x = e^u$, where $u = \ln x$. Then $\dfrac{dx}{du} = e^u$. Hence

$$\begin{aligned}\int \frac{\ln x}{x}\,dx &= \int \frac{\ln(e^u)e^u}{e^u}\,du\\ &= \int u\,du\\ &= \tfrac{1}{2}u^2 + c\\ &= \tfrac{1}{2}(\ln x)^2 + c.\end{aligned}$$

(d) Let $x = \tan u$, where $u = \arctan x$. Then $\dfrac{dx}{du} = \sec^2 u$. Hence

$$\begin{aligned}\int \frac{1}{(1+x^2)^{3/2}}\,dx &= \int \frac{1}{(1+\tan^2 u)^{3/2}} \times \sec^2 u\,du \\ &= \int \frac{\sec^2 u}{\sec^3 u}\,du \\ &= \int \frac{1}{\sec u}\,du \\ &= \int \cos u\,du \\ &= \sin u + c \\ &= \sin(\arctan x) + c.\end{aligned}$$

(Since $\tan u = x/1$, we have $\sin u = x/\sqrt{1+x^2}$, so this indefinite integral can be written as $x/\sqrt{1+x^2} + c$.)

Solution 3.4

(a) Let $u = x^5$ so $\dfrac{du}{dx} = 5x^4$. Hence

$$\begin{aligned}\int x^4 \sin(x^5)\,dx &= \tfrac{1}{5}\int \sin(x^5)(5x^4)\,dx \\ &= \tfrac{1}{5}\int \sin u\,du \\ &= \tfrac{1}{5}(-\cos u) + c \\ &= -\tfrac{1}{5}\cos(x^5) + c.\end{aligned}$$

(b) Let $u = \cos(2x)$ so $\dfrac{du}{dx} = -2\sin(2x)$. Hence

$$\begin{aligned}\int \cos^7(2x)\sin(2x)\,dx &= -\tfrac{1}{2}\int \cos^7(2x)(-2\sin(2x))\,dx \\ &= -\tfrac{1}{2}\int u^7\,du \\ &= -\tfrac{1}{2} \times \tfrac{1}{8}u^8 + c \\ &= -\tfrac{1}{16}\cos^8(2x) + c.\end{aligned}$$

(c) Let $u = x - 1$ so $\dfrac{du}{dx} = 1$ and $x = u + 1$. Hence

$$\begin{aligned}\int x\sqrt{x-1}\,dx &= \int (u+1)u^{1/2}\,du \\ &= \int (u^{3/2} + u^{1/2})\,du \\ &= \tfrac{2}{5}u^{5/2} + \tfrac{2}{3}u^{3/2} + c \\ &= \tfrac{2}{5}(x-1)^{5/2} + \tfrac{2}{3}(x-1)^{3/2} + c.\end{aligned}$$

(d) Let $u = e^x$ so $\dfrac{du}{dx} = e^x$. Hence

$$\begin{aligned}\int e^x \cos(e^x)\,dx &= \int \cos(e^x)(e^x)\,dx \\ &= \int \cos u\,du \\ &= \sin u + c \\ &= \sin(e^x) + c.\end{aligned}$$

Solution 3.5

(a) Use integration by substitution.

Let $u = 4x^3$ so $\dfrac{du}{dx} = 12x^2$. Hence

$$\begin{aligned}\int x^2 \sin(4x^3)\,dx &= \tfrac{1}{12}\int \sin(4x^3)(12x^2)\,dx \\ &= \tfrac{1}{12}\int \sin u\,du \\ &= -\tfrac{1}{12}\cos u + c \\ &= -\tfrac{1}{12}\cos(4x^3) + c.\end{aligned}$$

(b) Use integration by parts.

Let $f(x) = x$ and $g'(x) = \sin(6x)\,dx$; then

$$f'(x) = 1 \quad \text{and} \quad g(x) = -\tfrac{1}{6}\cos(6x).$$

Hence

$$\begin{aligned}\int x\sin(6x)\,dx &= -\tfrac{1}{6}x\cos(6x) - \int \left(-\tfrac{1}{6}\cos(6x)\right)dx \\ &= -\tfrac{1}{6}x\cos(6x) + \tfrac{1}{6} \times \tfrac{1}{6}\sin(6x) + c \\ &= \tfrac{1}{36}\sin(6x) - \tfrac{1}{6}x\cos(6x) + c.\end{aligned}$$

(c) Use integration by substitution.

Let $u = 8 - x^4$ so $\dfrac{du}{dx} = -4x^3$. Hence

$$\begin{aligned}\int \frac{x^3}{(8-x^4)^6}\,dx &= -\tfrac{1}{4}\int \frac{1}{(8-x^4)^6}(-4x^3)\,dx \\ &= -\tfrac{1}{4}\int \frac{1}{u^6}\,du \\ &= -\tfrac{1}{4}\left(-\tfrac{1}{5} \times \frac{1}{u^5}\right) + c \\ &= \frac{1}{20(8-x^4)^5} + c.\end{aligned}$$

Solution 3.6

(a) Use integration by substitution.

Let $u = x + 1$ so $\dfrac{du}{dx} = 1$.

Also $u = 2$ when $x = 1$, and $u = 3$ when $x = 2$. Hence

$$\begin{aligned}\int_1^2 \frac{x+2}{(x+1)^2}\,dx &= \int_2^3 \frac{u+1}{u^2}\,du \\ &= \int_2^3 \left(\frac{1}{u} + \frac{1}{u^2}\right) du \\ &= \left[\ln|u| - \frac{1}{u}\right]_2^3 \\ &= \left(\ln 3 - \tfrac{1}{3}\right) - \left(\ln 2 - \tfrac{1}{2}\right) \\ &= \ln\tfrac{3}{2} + \tfrac{1}{6} \\ &= 0.5721 \text{ (to 4 d.p.).}\end{aligned}$$

(b) Use integration by substitution.

Let $u = \sec x$ so $\dfrac{du}{dx} = \sec x \tan x$.

Also $u = 1$ when $x = 0$, and $u = \dfrac{1}{\cos(\pi/4)} = \sqrt{2}$ when $x = \pi/4$. Hence

$$\begin{aligned}\int_0^{\pi/4} \sec^4 x \tan x\,dx &= \int_0^{\pi/4} \sec^3 x(\sec x \tan x)\,dx \\ &= \int_1^{\sqrt{2}} u^3\,du = \left[\tfrac{1}{4}u^4\right]_1^{\sqrt{2}} \\ &= \tfrac{1}{4}(4-1) = 0.75.\end{aligned}$$

(c) Use integration by parts.

Let $f(x) = x$ and $g'(x) = \cos(6x)$; then

$$f'(x) = 1 \quad \text{and} \quad g(x) = \tfrac{1}{6}\sin(6x).$$

Hence

$$\begin{aligned}\int_{-\pi/6}^{\pi/6} x\cos(6x)\,dx &= \left[x\left(\tfrac{1}{6}\sin(6x)\right)\right]_{-\pi/6}^{\pi/6} \\ &\quad - \tfrac{1}{6}\int_{-\pi/6}^{\pi/6} \sin(6x)\,dx \\ &= \tfrac{\pi}{36}\sin\pi + \tfrac{\pi}{36}\sin(-\pi) \\ &\quad + \tfrac{1}{36}\Big[\cos(6x)\Big]_{-\pi/6}^{\pi/6} \\ &= \tfrac{1}{36}\Big(\cos\pi - \cos(-\pi)\Big) \\ &= \tfrac{1}{36}(-1-(-1)) = 0.\end{aligned}$$

Solution 4.1

(a) The volume is

$$\begin{aligned}\pi\int_{1/2}^1 \left(\frac{1}{x}\right)^2 dx &= \pi\int_{1/2}^1 x^{-2}\,dx \\ &= \pi\left[-\frac{1}{x}\right]_{1/2}^1 \\ &= \pi[-1+2] \\ &= \pi.\end{aligned}$$

(b) The ellipse is symmetrical about the x-axis, and the part of the ellipse above the x-axis is the curve

$$y = 2\sqrt{1 - \frac{x^2}{9}}$$

from $x = -3$ to $x = 3$. So the volume is given by

$$\begin{aligned}4\pi\int_{-3}^3 \left(1 - \frac{x^2}{9}\right) dx &= 4\pi\left[x - \frac{x^3}{27}\right]_{-3}^3 \\ &= 4\pi\left((3-1) - (-3+1)\right) \\ &= 16\pi.\end{aligned}$$

(c) The volume is

$$\pi\int_0^1 \left(\sqrt{x}(1+x)^{1/3}\right)^2 dx = \pi\int_0^1 x(1+x)^{2/3}\,dx.$$

Let $u = 1 + x$ so $\dfrac{du}{dx} = 1$ and $x = u - 1$.

Also $u = 1$ when $x = 0$, and $u = 2$ when $x = 1$. Hence

$$\begin{aligned}\pi\int_0^1 x(1+x)^{2/3}\,dx &= \pi\int_1^2 (u-1)u^{2/3}\,du \\ &= \pi\int_1^2 \left(u^{5/3} - u^{2/3}\right) du \\ &= \pi\left[\tfrac{3}{8}u^{8/3} - \tfrac{3}{5}u^{5/3}\right]_1^2 \\ &= \frac{3\pi}{40}\left[u^{5/3}(5u-8)\right]_1^2 \\ &= \frac{3\pi}{40}\left(2^{5/3}(10-8) - 1(5-8)\right) \\ &= \frac{3\pi}{40}\left(2^{8/3}+3\right) \\ &= 2.2029 \text{ (to 4 d.p.).}\end{aligned}$$

(d) The volume is

$$\pi\int_1^2 x^2 e^{-2x}\,dx.$$

Let $f(x) = x^2$ and $g'(x) = e^{-2x}$; then

$$f'(x) = 2x \quad \text{and} \quad g(x) = -\tfrac{1}{2}e^{-2x}.$$

Hence

$$\pi \int_1^2 x^2 e^{-2x}\,dx$$

$$= \pi\left(\left[x^2\left(-\tfrac{1}{2}e^{-2x}\right)\right]_1^2 - \int_1^2 2x\left(-\tfrac{1}{2}e^{-2x}\right)dx\right)$$

$$= \pi\left(-2e^{-4} + \tfrac{1}{2}e^{-2} + \int_1^2 xe^{-2x}\,dx\right). \qquad (1)$$

Now let $f(x) = x$ and $g'(x) = e^{-2x}$; then

$$f'(x) = 1 \quad\text{and}\quad g(x) = -\tfrac{1}{2}e^{-2x}.$$

Hence

$$\int_1^2 xe^{-2x}\,dx = \left[-\tfrac{1}{2}xe^{-2x}\right]_1^2 - \int_1^2 -\tfrac{1}{2}e^{-2x}\,dx$$

$$= -e^{-4} + \tfrac{1}{2}e^{-2} + \left[\tfrac{1}{2}\left(-\tfrac{1}{2}e^{-2x}\right)\right]_1^2$$

$$= -e^{-4} + \tfrac{1}{2}e^{-2} - \tfrac{1}{4}e^{-4} + \tfrac{1}{4}e^{-2}.$$

Substituting in equation (1), we obtain

$$\pi \int_1^2 x^2 e^{-2x}\,dx$$

$$= \pi\left(-2e^{-4} + \tfrac{1}{2}e^{-2} - e^{-4} + \tfrac{1}{2}e^{-2} - \tfrac{1}{4}e^{-4} + \tfrac{1}{4}e^{-2}\right)$$

$$= \pi\left(-\tfrac{13}{4}e^{-4} + \tfrac{5}{4}e^{-2}\right)$$

$$= 0.3445 \text{ (to 4 d.p.).}$$

(e) The volume is

$$\pi \int_0^{\pi} \sin^2 x\,dx.$$

Now $1 - 2\sin^2 x = \cos(2x)$, so

$$\sin^2 x = \tfrac{1}{2}(1 - \cos(2x)).$$

Hence

$$\pi \int_0^{\pi} \sin^2 x\,dx = \tfrac{1}{2}\pi \int_0^{\pi} (1 - \cos(2x))\,dx$$

$$= \tfrac{1}{2}\pi\left[x - \tfrac{1}{2}\sin(2x)\right]_0^{\pi}$$

$$= \tfrac{1}{2}\pi\Big((\pi - 0) - (0 - 0)\Big)$$

$$= \tfrac{1}{2}\pi^2$$

$$= 4.9348 \text{ (to 4 d.p.).}$$

Solution 4.2

Since the bottle is of uniform thickness 0.5 cm, including the base, the equation of the inside curve is $y = 1 + \frac{1}{\sqrt{2}}\cos x$, where $0 \le x \le 4$.

The volume of the inside of the bottle is therefore given by the volume of revolution generated by rotating the curve $y = 1 + \frac{1}{\sqrt{2}}\cos x$ about the x-axis between $x = 0$ and $x = 4$.

So

$$V = \pi \int_0^4 \left(1 + \tfrac{1}{\sqrt{2}}\cos x\right)^2 dx$$

$$= \pi \int_0^4 \left(1 + \sqrt{2}\cos x + \tfrac{1}{2}\cos^2 x\right) dx.$$

Now $2\cos^2 x - 1 = \cos(2x)$, so

$$\cos^2 x = \tfrac{1}{2}(1 + \cos(2x)).$$

Hence

$$V = \pi \int_0^4 \left(1 + \sqrt{2}\cos x + \tfrac{1}{4}(1 + \cos(2x))\right) dx$$

$$= \pi\left[x + \sqrt{2}\sin x + \tfrac{1}{4}x + \tfrac{1}{8}\sin(2x)\right]_0^4$$

$$= \pi\left(4 + \sqrt{2}\sin(4) + \tfrac{1}{4} \times 4 + \tfrac{1}{8}\sin(8) - 0\right)$$

$$= \pi\left(5 + \sqrt{2}\sin(4) + \tfrac{1}{8}\sin(8)\right)$$

$$= 12.73\,\text{cm}^3,$$

to the nearest $0.01\,\text{cm}^3$.

Solutions for Chapter C3

Solution 1.1

(a) Here $f(x) = e^{-x}$, so $(0, f(0)) = (0, e^0) = (0, 1)$. Also $f'(x) = -e^{-x}$, so the gradient of the curve at $(0, 1)$ is $f'(0) = -1$. Thus the required line passes through the point $(0, 1)$ and has gradient -1. Its equation is therefore $y = 1 - x$.

So the linear Taylor polynomial is $p(x) = 1 - x$.

Taking $x = 0.02$, we obtain $p(0.02) = 0.98$.

To eight decimal places, the remainder is

$$\begin{aligned} r(0.02) &= e^{-0.02} - 0.98 \\ &= 0.000\,198\,67. \end{aligned}$$

(b) Here $f(x) = (4 - x)^{1/2}$, so $(0, f(0)) = (0, 2)$. Also
$f'(x) = -\frac{1}{2}(4 - x)^{-1/2}$, so the gradient of the curve at $(0, 2)$ is $-\frac{1}{2} \times 4^{-1/2} = -\frac{1}{4}$.

Thus the required line passes through the point $(0, 2)$ and has gradient $-\frac{1}{4}$. Its equation is therefore $y = 2 - \frac{1}{4}x$.

So the linear Taylor polynomial is $p(x) = 2 - \frac{1}{4}x$.

Taking $x = 0.02$, we obtain $p(0.02) = 1.995$.

To eight decimal places, the remainder is

$$\begin{aligned} r(0.02) &= (4 - 0.02)^{1/2} - 1.995 \\ &= -0.000\,006\,27. \end{aligned}$$

Solution 1.2

(a) Here $f(x) = e^{-x}$, so $(1, f(1)) = (1, 1/e)$. Also $f'(x) = -e^{-x}$, so $f'(1) = -1/e$. Thus the required line has gradient $-1/e$. Its equation is therefore $y = a_0 - x/e$ and it passes through $(1, 1/e)$. Hence $1/e = a_0 - 1/e$, so $a_0 = 2/e$ and the equation of the line is $y = 2/e - x/e$. The linear Taylor polynomial about 1 is therefore

$$p(x) = \frac{2}{e} - \frac{x}{e}.$$

(b) Here $f(x) = \dfrac{1}{1 + x}$, so $(1, f(1)) = (1, \frac{1}{2})$. Also
$f'(x) = -\dfrac{1}{(1 + x)^2}$, so $f'(1) = -\frac{1}{4}$. Thus the required line has gradient $-\frac{1}{4}$. Its equation is therefore $y = a_0 - \frac{1}{4}x$ and it passes through $(1, \frac{1}{2})$. Hence $\frac{1}{2} = a_0 - \frac{1}{4}$, so $a_0 = \frac{3}{4}$ and the equation of the line is $y = \frac{3}{4} - \frac{1}{4}x$. The linear Taylor polynomial about 1 is therefore

$$p(x) = \tfrac{3}{4} - \tfrac{1}{4}x.$$

Solution 1.3

(a) Solution 1.2(b) gave the linear Taylor polynomial about 1 for $f(x) = \dfrac{1}{1 + x}$ as $p(x) = \frac{3}{4} - \frac{1}{4}x$.

Taking $x = 1.01$ gives $f(1.01) = \dfrac{1}{2.01}$.

So the reciprocal of 2.01 is approximated by $p(1.01) = \frac{1}{4}(3 - 1.01) = 0.4975$.

To eight decimal places, the remainder is given by

$$r(1.01) = \frac{1}{2.01} - 0.4975 = 0.000\,012\,44.$$

(b) Here $f(x) = \dfrac{1}{(1 + x)^3}$, so $(0, f(0)) = (0, 1)$. Also $f'(x) = -\dfrac{3}{(1 + x)^4}$, so $f'(0) = -3$. Thus the required line has gradient -3 and passes through $(0, 1)$. Its equation is therefore $y = 1 - 3x$, and the linear Taylor polynomial about 0 is

$$p(x) = 1 - 3x.$$

Taking $x = 0.01$ gives $f(0.01) = \dfrac{1}{(1.01)^3}$.

So $1/(1.01)^3$ is approximated by

$$p(0.01) = 1 - 3 \times 0.01 = 0.97.$$

To eight decimal places, the remainder is given by

$$r(0.01) = \frac{1}{(1.01)^3} - 0.97 = 0.000\,590\,15.$$

Solution 1.4

(a) Let the polynomial we seek be

$$p(x) = a_0 + a_1 x + a_2 x^2.$$

First we ensure that $p(0) = f(0)$. We have

$$\begin{aligned} f(x) &= e^{x/2}, \quad p(x) = a_0 + a_1 x + a_2 x^2, \\ f(0) &= e^0 = 1, \quad p(0) = a_0. \end{aligned}$$

Thus we have $a_0 = 1$.

Next we ensure that $p'(0) = f'(0)$. We have

$$\begin{aligned} f'(x) &= \tfrac{1}{2}e^{x/2}, \quad p'(x) = a_1 + 2a_2 x, \\ f'(0) &= \tfrac{1}{2}e^0 = \tfrac{1}{2}, \quad p'(0) = a_1. \end{aligned}$$

Thus we have $a_1 = \frac{1}{2}$.

Finally we ensure that $p''(0) = f''(0)$. We have

$$\begin{aligned} f''(x) &= \tfrac{1}{4}e^{x/2}, \quad p''(x) = 2a_2, \\ f''(0) &= \tfrac{1}{4}e^0 = \tfrac{1}{4}, \quad p''(0) = 2a_2. \end{aligned}$$

Thus we have $a_2 = \frac{1}{8}$.

Hence the quadratic Taylor polynomial about 0 for $f(x) = e^{x/2}$ is

$$p(x) = 1 + \tfrac{1}{2}x + \tfrac{1}{8}x^2.$$

The polynomial p gives the approximation

$$\begin{aligned} f(0.01) \simeq p(0.01) &= 1 + \tfrac{1}{2}(0.01) + \tfrac{1}{8}(0.01)^2 \\ &= 1.005\,0125. \end{aligned}$$

The associated remainder, to eight decimal places, is

$$\begin{aligned} r(0.01) &= e^{0.01/2} - 1.005\,0125 \\ &= 0.000\,000\,02. \end{aligned}$$

(b) Let the polynomial we seek be

$$p(x) = a_0 + a_1x + a_2x^2.$$

First we ensure that $p(0) = f(0)$. We have

$$f(x) = x\cos x, \quad p(x) = a_0 + a_1x + a_2x^2,$$
$$f(0) = 0, \quad p(0) = a_0.$$

Thus we have $a_0 = 0$.

Next we ensure that $p'(0) = f'(0)$. We have

$$f'(x) = \cos x - x\sin x, \quad p'(x) = a_1 + 2a_2x,$$
$$f'(0) = 1, \quad p'(0) = a_1.$$

Thus we have $a_1 = 1$.

Finally we ensure that $p''(0) = f''(0)$. We have

$$f''(x) = -\sin x - (\sin x + x\cos x), \quad p''(x) = 2a_2,$$
$$f''(0) = 0, \quad p''(0) = 2a_2.$$

Thus we have $a_2 = 0$.

Hence the quadratic Taylor polynomial about 0 for $f(x) = x\cos x$ is

$$p(x) = x.$$

The polynomial p gives the approximation

$$f(0.01) \simeq p(0.01) = 0.01.$$

The associated remainder, to eight decimal places, is

$$\begin{aligned} r(0.01) &= 0.01\cos(0.01) - 0.01 \\ &= -0.000\,0005. \end{aligned}$$

Solution 2.1

To find the quartic Taylor polynomial about 0 for any function $f(x)$ we need to evaluate $f(0)$, $f'(0)$, $f''(0)$, $f^{(3)}(0)$ and $f^{(4)}(0)$.

(a) For $f(x) = \cos(2x)$, we have:

$$\begin{aligned} f(x) &= \cos(2x), & f(0) &= 1; \\ f'(x) &= -2\sin(2x), & f'(0) &= 0; \\ f''(x) &= -4\cos(2x), & f''(0) &= -4; \\ f^{(3)}(x) &= 8\sin(2x), & f^{(3)}(0) &= 0; \\ f^{(4)}(x) &= 16\cos(2x), & f^{(4)}(0) &= 16. \end{aligned}$$

The quartic Taylor polynomial about 0 for $\cos(2x)$ is therefore

$$\begin{aligned} p(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 \\ &\quad + \frac{f^{(4)}(0)}{4!}x^4 \\ &= 1 - \frac{4}{2!}x^2 + \frac{16}{4!}x^4 \\ &= 1 - 2x^2 + \tfrac{2}{3}x^4. \end{aligned}$$

(b) To make the differentiation easier, note that

$$f(x) = \ln\left(\frac{1}{1+x}\right) = -\ln(1+x).$$

Thus we have:

$$\begin{aligned} f(x) &= -\ln(1+x), & f(0) &= 0; \\ f'(x) &= -\frac{1}{1+x}, & f'(0) &= -1; \\ f''(x) &= \frac{1}{(1+x)^2}, & f''(0) &= 1; \\ f^{(3)}(x) &= \frac{-2}{(1+x)^3}, & f^{(3)}(0) &= -2; \\ f^{(4)}(x) &= \frac{6}{(1+x)^4}, & f^{(4)}(0) &= 6. \end{aligned}$$

The quartic Taylor polynomial about 0 for $\ln\left(\dfrac{1}{1+x}\right)$ is therefore

$$\begin{aligned} p(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 \\ &\quad + \frac{f^{(4)}}{4!}(0)x^4 \\ &= -x + \frac{1}{2!}x^2 - \frac{2}{3!}x^3 + \frac{6}{4!}x^4 \\ &= -x + \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 + \tfrac{1}{4}x^4. \end{aligned}$$

(c) For $f(x) = (1-x)^{1/2}$, we have:

$$\begin{aligned} f(x) &= (1-x)^{1/2}, & f(0) &= 1; \\ f'(x) &= -\tfrac{1}{2}(1-x)^{-1/2}, & f'(0) &= -\tfrac{1}{2}; \\ f''(x) &= -\tfrac{1}{4}(1-x)^{-3/2}, & f''(0) &= -\tfrac{1}{4}; \\ f^{(3)}(x) &= -\tfrac{3}{8}(1-x)^{-5/2}, & f^{(3)}(0) &= -\tfrac{3}{8}; \\ f^{(4)}(x) &= -\tfrac{15}{16}(1-x)^{-7/2}, & f^{(4)}(0) &= -\tfrac{15}{16}. \end{aligned}$$

The quartic Taylor polynomial about 0 for $(1-x)^{1/2}$ is therefore

$$\begin{aligned} p(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 \\ &\quad + \frac{f^{(4)}(0)}{4!}x^4 \\ &= 1 - \tfrac{1}{2}x + \frac{\left(-\frac{1}{4}\right)}{2!}x^2 + \frac{\left(-\frac{3}{8}\right)}{3!}x^3 + \frac{\left(-\frac{15}{16}\right)}{4!}x^4 \\ &= 1 - \tfrac{1}{2}x - \tfrac{1}{8}x^2 - \tfrac{1}{16}x^3 - \tfrac{5}{128}x^4. \end{aligned}$$

Solution 2.2

(a) The quintic Taylor polynomial about π for a function f is given by

$$f(\pi) + f'(\pi)(x-\pi) + \frac{f''(\pi)}{2!}(x-\pi)^2 + \cdots + \frac{f^{(5)}(\pi)}{5!}(x-\pi)^5.$$

We have

$$\begin{aligned} f(x) &= \cos x, & f(\pi) &= -1;\\ f'(x) &= -\sin x, & f'(\pi) &= 0;\\ f''(x) &= -\cos x, & f''(\pi) &= 1;\\ f^{(3)}(x) &= \sin x, & f^{(3)}(\pi) &= 0;\\ f^{(4)}(x) &= \cos x, & f^{(4)}(\pi) &= -1;\\ f^{(5)}(x) &= -\sin x, & f^{(5)}(\pi) &= 0. \end{aligned}$$

So the quintic Taylor polynomial about π for $\cos x$ is

$$p(x) = -1 + \tfrac{1}{2}(x-\pi)^2 - \tfrac{1}{24}(x-\pi)^4.$$

(b) The quintic Taylor polynomial about e for a function f is given by

$$f(e) + f'(e)(x-e) + \frac{f''(e)}{2!}(x-e)^2 + \cdots + \frac{f^{(5)}(e)}{5!}(x-e)^5.$$

We have

$$\begin{aligned} f(x) &= \ln x, & f(e) &= \ln e = 1;\\ f'(x) &= \frac{1}{x}, & f'(e) &= \frac{1}{e};\\ f''(x) &= -\frac{1}{x^2} & f''(e) &= -\frac{1}{e^2};\\ f^{(3)}(x) &= \frac{2}{x^3}, & f^{(3)}(e) &= \frac{2}{e^3};\\ f^{(4)}(x) &= -\frac{6}{x^4} & f^{(4)}(e) &= -\frac{6}{e^4};\\ f^{(5)}(x) &= \frac{24}{x^5}, & f^{(5)}(e) &= \frac{24}{e^5}. \end{aligned}$$

So the quintic Taylor polynomial about e for $\ln x$ is

$$\begin{aligned} p(x) &= 1 + \frac{1}{e}(x-e) - \frac{1}{e^2}\frac{(x-e)^2}{2!} + \frac{2}{e^3}\frac{(x-e)^3}{3!}\\ &\quad - \frac{6}{e^4}\frac{(x-e)^4}{4!} + \frac{24}{e^5}\frac{(x-e)^5}{5!}\\ &= 1 + \frac{1}{e}(x-e) - \frac{1}{2e^2}(x-e)^2 + \frac{1}{3e^3}(x-e)^3\\ &\quad - \frac{1}{4e^4}(x-e)^4 + \frac{1}{5e^5}(x-e)^5. \end{aligned}$$

Solution 2.3

(a) The Taylor polynomial of degree n about 0 for the function $f(x) = \ln(1+x)$ is

$$p_n(x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \cdots + \frac{(-1)^{n+1}}{n}x^n.$$

For $x = 0.05$, we obtain to six decimal places

$$\begin{aligned} p_1(0.05) &= 0.05,\\ p_2(0.05) &= p_1(0.05) - \tfrac{1}{2}(0.05)^2 = 0.048\,75,\\ p_3(0.05) &= p_2(0.05) + \tfrac{1}{3}(0.05)^3 = 0.048\,792,\\ p_4(0.05) &= p_3(0.05) - \tfrac{1}{4}(0.05)^4 = 0.048\,790,\\ p_5(0.05) &= p_4(0.05) + \tfrac{1}{5}(0.05)^5 = 0.048\,790. \end{aligned}$$

The values of $p_4(0.05)$ and $p_5(0.05)$ agree to six decimal places, so it is likely that to four decimal places

$$\ln(1.05) = 0.0488.$$

(This is the case.)

(b) The Taylor polynomial of degree $2n$ about 0 for the function $f(x) = \cos(2x)$ is

$$p_{2n}(x) = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \cdots + \frac{(-1)^n(2x)^{2n}}{(2n)!}.$$

For $x = 0.1$, we obtain to six decimal places

$$\begin{aligned} p_0(0.1) &= 1,\\ p_2(0.1) &= p_0(0.1) - \frac{(2\times 0.1)^2}{2!} = 0.98,\\ p_4(0.1) &= p_2(0.1) + \frac{(2\times 0.1)^4}{4!} = 0.980\,067,\\ p_6(0.1) &= p_4(0.1) - \frac{(2\times 0.1)^6}{6!} = 0.980\,067. \end{aligned}$$

The values of $p_4(0.1)$ and $p_6(0.1)$ agree to six decimal places, so it is likely that to four decimal places

$$\cos(0.2) = 0.9801.$$

(This is the case.)

Solution 3.1

(a) The Taylor series about 0 for a function f is given by

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots.$$

For $f(x) = e^{-x}$, we have:

$$\begin{aligned} f(x) &= e^{-x}, & f(0) &= e^0 = 1;\\ f'(x) &= -e^{-x}, & f'(0) &= -e^0 = -1;\\ f''(x) &= e^{-x}, & f''(0) &= e^0 = 1;\\ f^{(3)}(x) &= -e^{-x}, & f^{(3)}(0) &= -e^0 = -1;\\ f^{(n)}(x) &= (-1)^n e^{-x}, & f^{(n)}(0) &= (-1)^n. \end{aligned}$$

So the Taylor series about 0 for $f(x) = e^{-x}$ is

$$1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \cdots + \frac{(-1)^n}{n!}x^n + \cdots.$$

(b) The Taylor series about 1 for a function f is given by

$$f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f^{(3)}(1)}{3!}(x-1)^3 + \cdots + \frac{f^{(n)}(1)}{n!}(x-1)^n + \cdots.$$

For $f(x) = 1/x$ we have

$$\begin{aligned} f(x) &= 1/x, & f(1) &= 1;\\ f'(x) &= -\frac{1}{x^2}, & f'(1) &= -1;\\ f''(x) &= \frac{2}{x^3}, & f''(1) &= 2;\\ f^{(3)}(x) &= -\frac{6}{x^4}, & f^{(3)}(1) &= -6;\\ f^{(n)}(x) &= (-1)^n\frac{n!}{x^{n+1}}, & f^{(n)}(1) &= (-1)^n n! \end{aligned}$$

So the Taylor series about 1 for $f(x) = 1/x$ is

$$\begin{aligned} &1 - (x-1) + \frac{2}{2!}(x-1)^2 - \frac{6}{3!}(x-1)^3 + \cdots + \frac{(-1)^n n!}{n!}(x-1)^n + \cdots\\ &= 1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots + (-1)^n(x-1)^n + \cdots. \end{aligned}$$

Solution 3.2

The binomial series is given by

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots, \quad \text{for } -1 < x < 1.$$

(a) Thus the first five terms of the binomial series for $(1+x)^{-4}$ are

$$\begin{aligned} &1 + (-4)x + \frac{(-4)(-5)}{2!}x^2 + \frac{(-4)(-5)(-6)}{3!}x^3 + \frac{(-4)(-5)(-6)(-7)}{4!}x^4\\ &= 1 - 4x + 10x^2 - 20x^3 + 35x^4. \end{aligned}$$

(b) The first five terms of the binomial series for $(1+x)^{1/3}$ are

$$\begin{aligned} &1 + (\tfrac{1}{3})x + \frac{(\frac{1}{3})(-\frac{2}{3})}{2!}x^2 + \frac{(\frac{1}{3})(-\frac{2}{3})(-\frac{5}{3})}{3!}x^3 + \frac{(\frac{1}{3})(-\frac{2}{3})(-\frac{5}{3})(-\frac{8}{3})}{4!}x^4\\ &= 1 + \tfrac{1}{3}x - \tfrac{1}{9}x^2 + \tfrac{5}{81}x^3 - \tfrac{10}{243}x^4. \end{aligned}$$

Solution 3.3

(a) Use

$$(1+x)^{-4} = 1 - 4x + 10x^2 - 20x^3 + 35x^4 - \cdots.$$

Set $1 + x = 0.99$, so $x = -0.01$. To five decimal places, we have

$$\begin{aligned} p_0(-0.01) &= 1,\\ p_1(-0.01) &= p_0(-0.01) - 4(-0.01) = 1.04,\\ p_2(-0.01) &= p_1(-0.01) + 10(-0.01)^2 = 1.041,\\ p_3(-0.01) &= p_2(-0.01) - 20(-0.01)^3 = 1.041\,02,\\ p_4(-0.01) &= p_3(-0.01) + 35(-0.01)^4 = 1.041\,02. \end{aligned}$$

So it is likely that to three decimal places

$$(0.99)^{-4} = 1.041.$$

(This is the case.)

(b) Use

$$(1+x)^{1/3} = 1 + \tfrac{1}{3}x - \tfrac{1}{9}x^2 + \tfrac{5}{81}x^3 - \tfrac{10}{243}x^4 + \cdots.$$

Set $1 + x = 0.9$, so $x = -0.1$. To five decimal places, we have

$$\begin{aligned} p_0(-0.1) &= 1\\ p_1(-0.1) &= p_0(-0.1) + \tfrac{1}{3}(-0.1) = 0.966\,67,\\ p_2(-0.1) &= p_1(-0.1) - \tfrac{1}{9}(-0.1)^2 = 0.965\,56,\\ p_3(-0.1) &= p_2(-0.1) + \tfrac{5}{81}(-0.1)^3 = 0.965\,49,\\ p_4(-0.1) &= p_3(-0.1) - \tfrac{10}{243}(-0.1)^4 = 0.965\,49. \end{aligned}$$

So it is likely that to three decimal places

$$(0.9)^{1/3} = 0.965.$$

(This is the case.)

Solution 4.1

(a) The Taylor series about 0 for $\sin x$ is

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots, \quad \text{for } x \in \mathbb{R}.$$

Using this series, and replacing x by x^2 throughout, we obtain

$$\sin(x^2) = (x^2) - \frac{1}{3!}(x^2)^3 + \frac{1}{5!}(x^2)^5 - \frac{1}{7!}(x^2)^7 + \cdots.$$

The series is valid for any $x^2 \in \mathbb{R}$, so is valid for any $x \in \mathbb{R}$. Hence

$$\sin(x^2) = x^2 - \frac{1}{3!}x^6 + \frac{1}{5!}x^{10} - \frac{1}{7!}x^{14} + \cdots, \quad \text{for } x \in \mathbb{R}.$$

(b) The Taylor series about 0 for e^x is

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots, \quad \text{for } x \in \mathbb{R}.$$

Using this series, and replacing x by $x/2$, we obtain

$$e^{x/2} = 1 + \left(\frac{x}{2}\right) + \frac{1}{2!}\left(\frac{x}{2}\right)^2 + \frac{1}{3!}\left(\frac{x}{2}\right)^3 + \cdots.$$

The series is valid for $\frac{1}{2}x \in \mathbb{R}$, so it is valid for $x \in \mathbb{R}$. Hence

$$e^{x/2} = 1 + \tfrac{1}{2}x + \tfrac{1}{8}x^2 + \tfrac{1}{48}x^3 + \cdots, \text{ for } x \in \mathbb{R}.$$

(c) The Taylor series about 0 for $\dfrac{1}{1-x}$ is

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots.$$

Using this series, and replacing x by $2x$, we obtain

$$\frac{1}{1-2x} = 1 + (2x) + (2x)^2 + (2x)^3 + \cdots.$$

Since the Taylor series about 0 for $\dfrac{1}{1-x}$ is valid for $-1 < x < 1$, the above series is valid for $-1 < 2x < 1$; that is, for $-\frac{1}{2} < x < \frac{1}{2}$. Hence

$$\frac{1}{1-2x} = 1 + 2x + 4x^2 + 8x^3 + \cdots, \text{ for } -\tfrac{1}{2} < x < \tfrac{1}{2}.$$

(d) The Taylor series about 0 for $(1+x)^\alpha$ is

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots.$$

Using this series, replacing x by x^2, and putting $\alpha = \frac{3}{2}$, we obtain

$$(1+x^2)^{3/2} = 1 + \tfrac{3}{2}(x^2) + \tfrac{3}{2} \times \tfrac{1}{2} \times \frac{1}{2!}(x^2)^2 + \tfrac{3}{2} \times \tfrac{1}{2} \times \left(-\tfrac{1}{2}\right) \times \frac{1}{3!}(x^2)^3 + \cdots.$$

Since the Taylor series about 0 for $(1+x)^{3/2}$ is valid for $-1 < x < 1$, the above series is valid for $-1 < x^2 < 1$; that is, for $-1 < x < 1$. Hence

$$(1+x^2)^{3/2} = 1 + \tfrac{3}{2}x^2 + \tfrac{3}{8}x^4 - \tfrac{1}{16}x^6 + \cdots,$$

for $-1 < x < 1$.

Solution 4.2

(a) We have

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots,$$

for $x \in \mathbb{R}$, and

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots,$$

for $x \in \mathbb{R}$. Hence

$$e^x + \cos x = 2 + x + \tfrac{1}{6}x^3 + \tfrac{1}{12}x^4 + \cdots,$$

for $x \in \mathbb{R}$.

(b) We have

$$\ln\left(\frac{1-x}{1+x}\right) = \ln(1-x) - \ln(1+x).$$

Now the Taylor series about 0 for $\ln(1+x)$ is

$$\ln(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \cdots,$$

for $-1 < x < 1$.

Replacing x by $-x$ in this series gives

$$\ln(1-x) = -x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 \cdots.$$

Since the Taylor series about 0 for $\ln(1+x)$ is valid for $-1 < x < 1$, the above series is valid for $-1 < -x < 1$; that is, for $-1 < x < 1$. Hence

$$\begin{aligned}\ln\left(\frac{1-x}{1+x}\right) &= \left(-x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 - \tfrac{1}{5}x^5 - \cdots\right)\\ &\quad - \left(x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \tfrac{1}{5}x^5 - \cdots\right)\\ &= -2x - \tfrac{2}{3}x^3 - \tfrac{2}{5}x^5 - \tfrac{2}{7}x^7 - \cdots,\end{aligned}$$

for $-1 < x < 1$.

(c) We have

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots,$$

for $x \in \mathbb{R}$.

Multiplying both sides of the above equation by $1 - x$ gives

$$\begin{aligned}(1-x)\sin x &= \sin x - x\sin x\\ &= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)\\ &\quad - \left(x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \frac{x^8}{7!} \cdots\right),\end{aligned}$$

for $x \in \mathbb{R}$. Hence

$$(1-x)\sin x = x - x^2 - \frac{1}{3!}x^3 + \frac{1}{3!}x^4 + \cdots$$

for $x \in \mathbb{R}$.

(d) We have $(1+x)^2 = 1 + 2x + x^2$, for $x \in \mathbb{R}$, and

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots, \quad \text{for } x \in \mathbb{R}.$$

So

$$\begin{aligned}&(1+x)^2 e^x\\ &= (1 + 2x + x^2)\left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots\right),\\ &= \left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots\right)\\ &\quad \left(+ 2x + 2x^2 + \frac{2}{2!}x^3 + \cdots\right)\\ &\quad + \left(x^2 + x^3 + \cdots\right),\\ &= 1 + 3x + \tfrac{7}{2}x^2 + \tfrac{13}{6}x^3 + \cdots,\end{aligned}$$

for $x \in \mathbb{R}$. Hence

$$(1+x)^2 e^x = 1 + 3x + \tfrac{7}{2}x^2 + \tfrac{13}{6}x^3 + \cdots,$$

for $x \in \mathbb{R}$.

Solution 4.3

(a) The Taylor series about 0 for e^x is

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots, \text{ for } x \in \mathbb{R}.$$

Replacing x by $2x$ we obtain the Taylor series about 0 for e^{2x}:

$$\begin{aligned} e^{2x} &= 1 + (2x) + \frac{1}{2!}(2x)^2 + \frac{1}{3!}(2x)^3 + \cdots \\ &= 1 + 2x + \frac{4}{2!}x^2 + \frac{8}{3!}x^3 + \cdots, \end{aligned}$$

for $x \in \mathbb{R}$.

(b) Since $e^{2x} = e^x \times e^x$ we can obtain the Taylor series about 0 for e^{2x} by considering

$$\begin{aligned} e^x \times e^x &= \left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots\right) \\ &\quad \times \left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots\right) \\ &= \left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots\right) \\ &\quad + x\left(1 + x + \frac{1}{2!}x^2 + \cdots\right) \\ &\quad + \frac{1}{2!}x^2\left(1 + x + \cdots\right) + \frac{1}{3!}x^3(1 + \cdots) + \cdots \\ &= \left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots\right) \\ &\quad + \left(x + x^2 + \frac{1}{2!}x^3 + \cdots\right) \\ &\quad + \left(\frac{1}{2!}x^2 + \frac{1}{2!}x^3 + \cdots\right) + \frac{1}{3!}x^3 + \cdots \\ &= 1 + 2x + \frac{4}{2!}x^2 + \frac{8}{3!}x^3 + \cdots, \end{aligned}$$

for $x \in \mathbb{R}$.

This answer agrees with the solution in part (a).

Solution 4.4

(a) Using the given Taylor series,

$$\sinh x = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots, \text{ for } x \in \mathbb{R},$$

and replacing x by $2x$, we obtain the Taylor series for $\sinh(2x)$:

$$\sinh(2x) = (2x) + \frac{1}{3!}(2x)^3 + \frac{1}{5!}(2x)^5 + \cdots,$$

for $x \in \mathbb{R}$; that is,

$$\sinh(2x) = 2x + \tfrac{4}{3}x^3 + \tfrac{4}{15}x^5 + \cdots,$$

for $x \in \mathbb{R}$.

(b) Multiplication of the Taylor series for $\sinh x$ and $\cosh x$ gives

$$\begin{aligned} \sinh x \cosh x &= \left(x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots\right) \\ &\quad \times \left(1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots\right) \\ &= x\left(1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots\right) \\ &\quad + \frac{1}{3!}x^3\left(1 + \frac{1}{2!}x^2 + \cdots\right) \\ &\quad + \frac{1}{5!}x^5\left(1 + \cdots\right) + \cdots \\ &= x + \left(\frac{1}{2!} + \frac{1}{3!}\right)x^3 + \left(\frac{1}{4!} + \frac{1}{3!2!} + \frac{1}{5!}\right)x^5 \\ &\quad + \cdots \\ &= x + \tfrac{2}{3}x^3 + \tfrac{2}{15}x^5 + \cdots, \end{aligned}$$

for $x \in \mathbb{R}$. Hence

$$\sinh x \cosh x = x + \tfrac{2}{3}x^3 + \tfrac{2}{15}x^5 + \cdots, \text{ for } x \in \mathbb{R}.$$

(c) Since the Taylor series about 0 for $\sinh(2x)$ can be written (see part (a)) as

$$\sinh(2x) = 2\left(x + \tfrac{2}{3}x^3 + \tfrac{2}{15}x^5 + \ldots\right), \text{ for } x \in \mathbb{R},$$

we can conjecture that

$$\sinh(2x) = 2 \sinh x \cosh x.$$

(This identity is valid.)

Solution 4.5

(a) We have

$$\ln(1 + x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \cdots,$$

for $-1 < x < 1$.

Differentiating both sides gives

$$\frac{1}{1+x} = 1 - \tfrac{1}{2} \times (2x) + \tfrac{1}{3} \times (3x^2) - \tfrac{1}{4} \times (4x^3) + \cdots.$$

Hence

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots, \text{ for } -1 < x < 1,$$

which agrees with the standard Taylor series given in the chapter.

(b) Replacing x by $4x^2$ in the above series for $\dfrac{1}{1+x}$ gives

$$\frac{1}{1+4x^2} = 1 - (4x^2) + (4x^2)^2 - (4x^2)^3 + \cdots,$$

for $-1 < 4x^2 < 1$. Now $4x^2 > -1$ for all real x, so the condition for validity reduces to $4x^2 < 1$; that is, $-1 < 2x < 1$ or equivalently $-\tfrac{1}{2} < x < \tfrac{1}{2}$.

Hence

$$\frac{1}{1+4x^2} = 1 - 4x^2 + 16x^4 - 64x^6 + \cdots,$$

for $-\tfrac{1}{2} < x < \tfrac{1}{2}$.

(c) We have

$$\arctan(2x) = 2\int \frac{1}{1+4x^2}\,dx$$
$$= 2\int (1 - 4x^2 + 16x^4 - 64x^6 + \cdots)\,dx,$$

for $-\frac{1}{2} < x < \frac{1}{2}$, by part (b). Hence

$$\arctan(2x) = c + 2\left(x - \tfrac{4}{3}x^3 + \tfrac{16}{5}x^5 - \tfrac{64}{7}x^7 + \cdots\right).$$

When $x = 0$ we have $\arctan(2x) = 0$, so $c = 0$.
Thus

$$\arctan(2x) = 2x - \tfrac{8}{3}x^3 + \tfrac{32}{5}x^5 - \tfrac{128}{7}x^7 + \cdots,$$

for $-\frac{1}{2} < x < \frac{1}{2}$.

Solution 4.6

(a) The binomial series for $(1+x)^\alpha$ is

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots,$$

for $-1 < x < 1$.

Setting $\alpha = -\frac{1}{2}$ and replacing x by $(-x^2)$ gives

$$\frac{1}{\sqrt{1-x^2}} = 1 + \left(-\tfrac{1}{2}\right)\left(-x^2\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(-x^2)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}(-x^2)^3 + \cdots,$$

for $-1 < -x^2 < 1$.

Now $-x^2 < 1$ for all x, so the condition for validity reduces to $-1 < -x^2$. This is equivalent to $x^2 < 1$ or equivalently $-1 < x < 1$. Hence

$$\frac{1}{\sqrt{1-x^2}} = 1 + \tfrac{1}{2}x^2 + \tfrac{3}{8}x^4 + \tfrac{5}{16}x^6 + \cdots,$$

for $-1 < x < 1$.

(b) We have

$$\arccos x = -\int \frac{1}{\sqrt{1-x^2}}\,dx.$$

So, by part (a),

$$\arccos x = -\int \left(1 + \tfrac{1}{2}x^2 + \tfrac{3}{8}x^4 + \tfrac{5}{16}x^6 + \cdots\right)dx$$
$$= c - \left(x + \tfrac{1}{6}x^3 + \tfrac{3}{40}x^5 + \tfrac{5}{112}x^7 + \cdots\right),$$

for $-1 < x < 1$.

When $x = 0$ we have $\arccos x = \frac{1}{2}\pi$, so $c = \frac{1}{2}\pi$.
Hence

$$\arccos x = \tfrac{1}{2}\pi - x - \tfrac{1}{6}x^3 - \tfrac{3}{40}x^5 - \tfrac{5}{112}x^7 - \cdots,$$

for $-1 < x < 1$.

Printed in the United Kingdom by
Lithmark Limited